Tropical Varieties

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The tropical semi-ring

In the tropical semi-ring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ the operations are

$\oplus$ maximum

$\odot$ addition

Two examples:

$$5 \odot (3 \oplus 2) = 8$$

$$5 \odot 3 \oplus 5 \odot 2 = 8$$

The neutral element for $\oplus$ is $-\infty$.
The neutral element for $\odot$ is 0.

Tropical polynomials

- A *tropical monomial* in n variables:

$$c \odot x^v = c \odot \underbrace{x_1 \odot \cdots \odot x_1}_{v_1 \text{ times}} \odot \cdots \odot \underbrace{x_n \odot \cdots \odot x_n}_{v_n \text{ times}}$$

  where $c \in \mathbb{R}$ and $v \in \mathbb{N}^n$.
- A *tropical polynomial* is a finite tropical sum of tropical monomials with different exponent vectors.
- Evaluating a tropical monomial with zero-coefficient in a point $\omega \in \mathbb{R}^n$:

$$x^v(\omega) = \underbrace{\omega_1 \odot \cdots \odot \omega_1}_{v_1 \text{ times}} \odot \cdots \odot \underbrace{\omega_n \odot \cdots \odot \omega_n}_{v_n \text{ times}} = v \cdot \omega$$

- To evaluate a tropical polynomial in $\omega \in \mathbb{R}^n$ we evaluate its terms and take the maximum.

Tropical polynomials

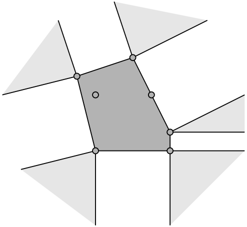
- Evaluating a polynomial f with zero coefficients in $\omega \in \mathbb{R}^n$ is equivalent to solving the following optimization problem:

$$\begin{aligned} &\text{maximize } \omega \cdot v \\ &\text{subject to } v \in NP(f) \end{aligned}$$

  where $NP(f)$ denotes the *Newton Polytope* of f.
- Tropical polynomial functions are piecewise linear.
- The regions of linearity are the outer normal cones of $NP(f)$.

Example

$$f = 0 \odot x_1^2 x_2 \oplus x_1^6 x_2 \oplus x_1^6 x_2^2 \oplus x_1^5 x_2^4 \oplus x_1^4 x_2^6 \oplus x_1 x_2^5 \oplus x_1^2 x_2^4$$



Tropical hypersurfaces

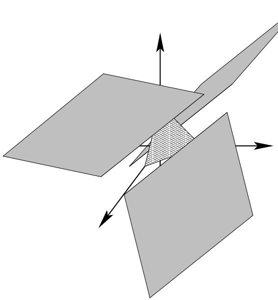
Define the zero-set of a tropical polynomial f in n-variables as:

$$T(f) = \{\omega \in \mathbb{R}^n : f(\omega) \text{ is a attained by at least two terms in } f\}$$

For zero-coefficient polynomials $T(f)$ is the union of all non-maximal cones in NF(NP(f)), where NF denotes the normal fan. The zero-set is also called a *tropical hypersurface*.

Example

$T(x_1 \oplus x_2 \oplus x_3) \subseteq \mathbb{R}^3$ is the union of three 2-dimensional cones:



“Zero-coefficients” is no restriction

Hypersurfaces of zero-coefficient polynomials are star-shaped.
But zero-coefficient are no real restriction:

$$p = (3) \oplus (2) \odot y \oplus (2) \odot x \oplus (0) \odot x \odot x$$

$$p(x, y) = \max(3, 2 + y, 2 + x, 2x)$$

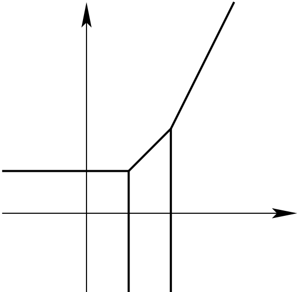
Question: When is the maximum attained twice?

$$q = t^3 \oplus t^2 \odot y \oplus t^2 \odot x \oplus x \odot x$$

$$q(x, y, t) = \max(3t, 2t + y, 2t + x, 2x)$$

Answer: When $q(x, y, t)$ is attained twice and $t = 1$.

$$T(p) \times \{1\} = T(q) \cap (\mathbb{R}^2 \times \{1\})$$



Tropicalization

- The tropicalization of a polynomial $f \in k[x_1, \ldots, x_n]$ is the tropical polynomial $\mathrm{trop}(f)$ where
  - $+$ and $\cdot$ have been changed to $\oplus$ and $\odot$.
  - The coefficients have been changed to 0.

Example

$$\mathrm{trop}(1x_1^2x_2 + 2x_1^6x_2 + 3x_1^6x_2^2 + 4x_1^5x_2^4 + 5x_1^4x_2^6 + 6x_1x_2^5 + 7x_1^2x_2^4) = 0 \odot x_1^2x_2 \oplus x_1^6x_2 \oplus x_1^6x_2^2 \oplus x_1^5x_2^4 \oplus x_1^4x_2^6 \oplus x_1x_2^5 \oplus x_1^2x_2^4$$

- For a field with a valuation we may consider an alternative definition where we take the valuation of the coefficients to get the tropical coefficients.

Tropical varieties

- For $f \in k[x_1, \ldots, x_n]$ we may write $T(f)$ for $T(\text{trop}(f))$.
- For $I \subseteq k[x_1, \ldots, x_n]$ the *tropical variety* of I is defined as

$$T(I) := \bigcap_{f \in I} T(f)$$

Lemma

For a principal ideal $\langle f \rangle \subseteq k[x_1, \ldots, x_n]$ we have

$$T(\langle f \rangle) = T(f).$$

Is every tropical variety a *finite* intersection of hypersurfaces?

Example: Grassmann 2,5

- Consider the 10 2x2 minors of a 2x5 matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \end{pmatrix} \qquad \begin{aligned} a &= x_{11}x_{22} - x_{12}x_{21} \\ b &= \dots \end{aligned}$$

- The five relations

$$\begin{aligned} bf - ah - ce &= 0 \\ bg - ai - de &= 0 \\ \dots \end{aligned}$$

  generate the Grassmann-Plücker ideal $G_{2,5}$.
- The tropical variety of $G_{2,5}$ is a subset of $\mathbb{R}^{10}$.
- It is the intersection of the hypersurfaces of the 5 relations.

To be continued...

Initial forms and initial ideals

Consider the polynomial ring $k[x_1, \ldots, x_n]$. Let $\omega \in \mathbb{R}^n$.

- The *weight* of a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with $a \in \mathbb{N}^n$ is $\langle \omega, a \rangle$.
- The *initial form* $in_\omega(f)$ of a polynomial $f \in k[x_1, \ldots, x_n]$ is the sum of terms with maximal weights.
  Example:
  $$in_{(1,2)}(x_1^4 + 2x_2^2 + x_1 x_2 + 1) = x_1^4 + 2x_2^2$$
- The *initial ideal* of an ideal $I \subseteq k[x_1, \ldots, x_n]$ is defined as
  $$in_\omega(I) = \langle in_\omega(f) \rangle_{f \in I}$$

Equivalent definition of tropical varieties

Observe: $T(f) = \{\omega \in \mathbb{R}^n : \mathrm{in}_\omega(f) \text{ is not a monomial}\}$.

Theorem

If $I \subseteq k[x_1, \ldots, x_n]$ is an ideal then

$$T(I) = \{\omega \in \mathbb{R}^n : \mathrm{in}_\omega(I) \text{ is monomial-free}\}$$

Proof.

We must prove

$$\bigcap_{f \in I} T(f) = \{\omega \in \mathbb{R}^n : \mathrm{in}_\omega(I) \text{ is monomial-free}\}$$

- $\supseteq$: Easy.
- $\subseteq$: A tiny bit more difficult...

□

The Gröbner fan of an ideal

Definition (Mora, Robbiano)

- Let $I \subseteq k[x_1, \ldots, x_n]$ be a homogeneous ideal.
- Define an equivalence relation $\sim$ on $\mathbb{R}^n$.

$$u \sim v \Leftrightarrow \mathrm{in}_u(I) = \mathrm{in}_v(I)$$

- The closure of each equivalence class is called a *Gröbner cone*.
- The set of all these cones is a polyhedral complex.
- We call this the *Gröbner fan* of I.

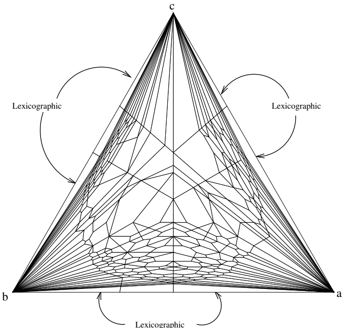
The Gröbner fan of an ideal

The following things are in bijection

- The full-dimensional Gröbner cones
- Monomial initial ideals of I
- The marked reduced Gröbner bases I

Example

$I = \langle a^5+b^3+c^2-1, a^2+b^2+c-1, a^6+b^5+c^3-1\rangle \subseteq \mathbb{Q}[a,b,c]$has 360 reduced Gröbner bases and 360 full-dimensional cones in its fan. (Not homogeneous!) Intersection of fan and 2-simplex:



A polyhedral structure on the tropical variety

- From the theorem/definition:

$$T(I) = \{\omega \in \mathbb{R}^n | in_\omega(I) \text{ is monomial-free}\}$$

  it is clear that $T(I)$ is a union of Gröbner cones.
- We may think of the tropical variety as a polyhedral complex inheriting its structure from the Gröbner fan.
- The tropical variety is a subcomplex of the Gröbner fan.

The homogeneity space

Definition

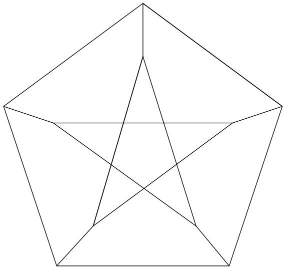
Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal.
The Gröbner cone equal to the equivalence class $\{\omega \in \mathbb{R}^n : \text{in}_\omega(I) = I\}$ is called the *homogeneity space* of I.

Lemma

- *The intersection of all Gröbner cones is the homogeneity space.*
- *The Gröbner cones are invariant under translation by vectors in the homogeneity space.*
- *The gradings for which I is homogeneous are exactly given by the vectors in the homogeneity space.*

Example: Grassmann 2,5 - continued

- The f-vector of the Gröbner fan of $G_{2,5}$ is $(1, 20, 120, 300, 330, 132)$.
- There is 1 five-dimensional cone (the homogeneity space).
- There are 132 ten-dimensional cones.
- Tropical variety is a 7-dimensional pure subcomplex.
- Modulo the homogeneity space it is 2-dimensional.
- Projectively we may draw it as a graph.



- The f-vector is $(1, 10, 15)$.
- The initial ideals for the the 15 seven-dimensional cones are all generated by binomials.

Tropical bases

Definition

Let C be Gröbner cone not contained in $T(I)$. A polynomial $f \in I$ is a *witness* for C if $T(f) \cap \operatorname{relint}(C) = \emptyset$.

Definition

A finite generating set F of I is a *tropical basis* if $T(I) = \bigcap_{f \in F} T(f)$

Theorem

Every ideal $I \subseteq k[x_1, \dots, x_n]$ *has a tropical basis.*

Proof.

Sketch of constructive proof.

- Start with a generating set of I.
- For every Gröbner cone C of I not contained in the tropical variety choose monomial in the initial ideal.
- "Lift" the monomial to a witness $f \in I$ for C.
- Add f to the generating set.

What if...

- the ideal I is not homogeneous?

Lemma

The initial ideal $\mathrm{in}_\omega(I)$ *contains a monomial if and only if* $\mathrm{in}_{(0,\omega)}({}^hI)$ *contains a monomial.*

- the ideal $I \subseteq \mathbb{C}\{\{t\}\}[x_1, \ldots, x_n]$ and we use the valuation to tropicalize the polynomial?

Lemma

If the ideal is generated by polynomials with coefficients in $\mathbb{C}[t]$ *just consider the ideal* I' *they generate in* $\mathbb{C}[x_1, \ldots, x_n, t]$*. The intersection of* $T(I')$ *with the* $t = 1$ *plane will be* $T(I) \times \{1\}$*.*

Don't forget the important theorem from Hannah's talk:

Theorem

- *Let* val : $(\mathbb{C}\{\{t\}\}^*)^n \to \mathbb{Q}^n$ *be coordinate-wise valuation.*
- *Let* $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ *be a homogeneous ideal and* $I' \subseteq \mathbb{C}\{\{t\}\}[x_1, \ldots, x_n]$ *the ideal it generates.*
- *The following identities hold:*

$$\mathrm{val}(V(I')) = T(I') \cap \mathbb{Q}^n = T(I) \cap \mathbb{Q}^n$$

Computing tropical varieties in Gfan

- The Gröbner fan approach gives us algorithms for computing tropical varieties.
- This requires further development of the techniques from the Gröbner walk [Collart, Kalkbrener, Mall].
- The algorithms are described in [Bogart, Jensen, Speyer, Sturmfels, Thomas]: “Computing tropical varieties”.
- The algorithms have been implemented in Gfan.
- Gfan will be presented in the workshop next week.