Algorithms in Discrete Morse Theory

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The Classical Set-up

Given a Morse function $f : M \to \mathbb{R}$ on a manifold $M$, we have the gradient vector field $\nabla f$ and the associated flow lines:
Recall that we can assign a number called the *index* to each critical point of $f$ in the following way. If $p \in M$ is a critical point, we can choose local coordinates at $p$ such that in some neighborhood, $f$ has the form

$$f = f(p) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2;$$

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On the torus, with $f$ the standard height function, the critical points $p, q, r, s$ have index 0, 1, 1, 2, respectively.
If we have such a function, we can figure out all sorts of things about $M$:

- $M$ has the homotopy type of a CW-complex with one cell of dimension $i$ for each critical point of index $i$
- The homology of $M$ can be calculated via this information

For example, here is a chain complex, built from the height function, computing the homology of the torus:

$$0 \to \mathbb{Z}[s] \xrightarrow{\partial = 0} \mathbb{Z}[q, r] \xrightarrow{\partial = 0} \mathbb{Z}[p] \to 0$$
Problem

Suppose $M$ is a finite simplicial complex (e.g. a triangulated manifold) and let $M_0$ be the set of vertices. Suppose we have a map

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Think: We have a sample of function values in some region. For example, $f$ may measure the temperature in a region in space or elevation data at certain longitudes and latitudes.
Question: Given only this information, can we construct a “gradient flow” on $M$ that mirrors the behavior of our function $f$?
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Answer: Yes, but it takes some work.
One Solution

Given a Morse function $f : M \to \mathbb{R}$, one can decompose $M$ into regions of uniform flow. This is the so-called *Morse–Smale complex*. Edelsbrunner, Harer, and Zomorodian ('03) developed a procedure to construct a combinatorial version of this complex on a PL 2-manifold from a sampled function. Knowledge of the Morse–Smale complex is equivalent to knowledge of the qualitative behavior of $f$. 
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Their algorithm works well, but it is limited. In fact, they were able to extend it to PL 3-manifolds, but it was very difficult. For higher dimensional manifolds, it’s not at all clear how one should proceed.
Our Idea

Use Forman’s discrete Morse theory to extend an arbitrary function $f : M_0 \to \mathbb{R}$ to a discrete Morse function $f : M \to \mathbb{R}$. 
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Our algorithm works in arbitrary dimensions, not just 2 and 3.
Discrete Morse Theory

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It must satisfy the following two conditions, for every $p$-simplex $\alpha^{(p)}$ in $M$:

1. $\#\{\beta^{(p+1)} > \alpha^{(p)} | f(\beta) \leq f(\alpha)\} \leq 1$;
2. $\#\{\tau^{(p-1)} < \alpha^{(p)} | f(\tau) \geq f(\alpha)\} \leq 1$. 
Think: Function values increase with the dimension of the simplices.
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Simple example: \( f : M \rightarrow \mathbb{R}, \ f(\alpha) = \dim \alpha \)
Critical Points

A simplex $\alpha^{(p)}$ is *critical* if the following two conditions hold:

1. $\#\{\beta^{(p+1)} > \alpha^{(p)} | f(\beta) \leq f(\alpha)\} = 0$;

2. $\#\{\tau^{(p-1)} < \alpha^{(p)} | f(\tau) \geq f(\alpha)\} = 0$.

That is, $\alpha$ is critical provided $f$ decreases when leaving $\alpha$ via a face, and $f$ increases when leaving $\alpha$ via a coface.
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A simplex that is not critical is called *regular*. 
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Examples

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2. Critical $n$-cell $=$ Local maximum
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2. Critical $n$-cell = Local maximum

3. For $f(\alpha) = \dim \alpha$, every cell is critical.
Here is a discrete Morse function on the circle:
Here is a discrete Morse function on the circle:

There are two critical cells, $f^{-1}(0)$ and $f^{-1}(5)$. 
Here is a discrete Morse function on the torus:
Here is a discrete Morse function on the torus:

![Discrete Morse function diagram]

The critical cells are $f^{-1}(0)$, $f^{-1}(42)$, $f^{-1}(44)$, and $f^{-1}(86)$. 
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So, the torus has the homotopy type of a complex with one vertex, two 1-cells, and one 2-cell.
The Associated Gradient Field

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To visualize this: draw an arrow

\[ \alpha^{(p)} \rightarrow \beta^{(p+1)} \]

for each such pair.

For any \( \sigma \) in \( M \), exactly one of the following is true:

1. \( \sigma \) is the tail of exactly one arrow;
2. \( \sigma \) is the head of exactly one arrow;
3. \( \sigma \) is neither the head nor tail of an arrow.
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2. $\sigma$ is the head of exactly one arrow;
3. $\sigma$ is neither the head nor tail of an arrow.

In the last case, $\sigma$ is critical.
Here is the gradient field on the torus associated to the above discrete Morse function:
The Algorithm

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**Theorem: (Forman)** \( f \) may be extended to a discrete Morse function on \( M \).

**Proof:** Let \( c = \max \{ f(x) : x \in M_0 \} \), and set \( f(\sigma) = c + \dim \sigma \) for each cell \( \sigma \) in \( M - M_0 \).
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This works, but it makes every cell not in \( M_0 \) critical, and so it’s not particularly useful.
Crucial Question:

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First, a definition:

If \( v \in M_0 \), the link of \( v \) is the simplicial complex \( L \) whose simplices are all \( \tau = [v_1, \ldots, v_r] \) such that \( v * \tau = [v, v_1, \ldots, v_r] \) is a simplex in \( M \).
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The lower link of $v$ is the maximal subcomplex of $L$ having all vertices with $f$-value less than that of $v$. 
Example
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The lower link of $v$ is shown in red.
For surfaces, we can define the *index* of a vertex to be:

\[
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\]

In this example, the index is 1.
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- index = 1 means that $v$ is an ordinary saddle (two directions up, two directions down)
- index = $-1$ means that $v$ is a minimum or a maximum
- index $\geq 2$ indicates that $v$ is a multiple saddle point (degenerate critical point)
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- a bijection \( r : B \to A \) so that \( r(\sigma) \) is a codimension-one face of \( \sigma \).
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$C \leftrightarrow$ critical simplices

$r : B \rightarrow A$ pairs regular simplices
Extract\((M, f, p)\)

- ExtractRaw\((M, f)\)
- for \(j = 1\) to \(\text{dim } M\)
  - ExtractCancel\((M, f, p, j)\)
- end for
Extract\((M, f, p)\)

- \textbf{ExtractRaw}(M, f)
- for \(j = 1\) to \(\dim M\)
  - \textbf{ExtractCancel}(M, f, p, j)
- end for

\textbf{ExtractCancel} is a procedure to cancel pairs of critical simplices that are joined by a single gradient path and whose values differ by at most \(p\) (the parameter \(p\) is called \textit{persistence}). Classical Morse theory tells us that we can do this without altering the other critical points.
• Initialize $A, B, C$ to be empty.

• foreach $v \in M_0$
  
  – let $M' = \text{the lower link of } v$
  
  – if $M'$ is empty then add $v$ to $C$  % local min
  
  – else
    
    * Add $v$ to $A$.
    * Let $f': M'_0 \rightarrow \mathbb{R}$ be the restriction of $f$.
    * Extract($M', f', \infty$) and let $A', B', C', r'$ denote the resulting partition of the simplices of $M'$
    * find the $w_0 \in C'_0$ so that $f'(w_0)$ is the smallest. Add $[v, w_0]$ to $B$ and define $r([v, w_0]) = v$.
    * for each $\sigma \in C' - w_0$ add $v * \sigma$ to $C$.
    * for each $\sigma \in B'$ add $v * \sigma$ to $B$, add $v * r'(\sigma)$ to $A$ and define $r(v * \sigma) = v * r'(\sigma)$.
  
  – end if

• continue foreach
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Theorem: The discrete vector field produced by Extract is the gradient field of a discrete Morse function on \( M \).

Does it mirror the behavior of \( f : M_0 \to \mathbb{R} \)?

Theorem:

1. There is an extension of \( f \) to a discrete Morse function \( f' : M \to \mathbb{R} \) with the same gradient field as that produced by Extract.

2. If \( \sigma \) is a simplex in \( M \), denote by \( \max f(\sigma) \) the maximum of all \( f(v) \) as \( v \) ranges over the vertices of \( \sigma \). Then, given \( \epsilon > 0 \) we may choose such an \( f' \) so that \( |f'(\tau) - \max f(\tau)| \leq \epsilon \) for any simplex \( \tau \).
Examples
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What happens if we run the algorithm on the torus?

The algorithm finds a single critical vertex at $f^{-1}(0)$, two critical edges at $f^{-1}(42)$ and $f^{-1}(44)$, and a single critical triangle at $f^{-1}(82)$. 
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This is not the critical triangle of the original discrete Morse function, but it is adjacent to the vertex with maximal value, and the average value of its vertices is greatest with this property.
Pilot Mountain, NC
Here’s the topographical map:
Let’s put a grid on it and measure the elevation at each grid point:
Head CT images
Renal Scintigrams