Globally hyperbolic Lorentzian manifolds with special holonomy

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Before I start with the topic in the title, I will explain how this topic is related to the subject of the Summer Program and to some other talks during the workshop

(Cooperation with I. Kath, F. Leitner, Th. Leistner, Th. Neukirchner, A. Galaev, O. Müller)

1. Relation to overdetermined systems of PDE and conformally invariant operators
   - Spinors on curved spaces
   - Conformally invariant operators on spinors
   - Conformal Killing spinors and special geometries
2. Holonomy of connections and parallel sections
   - General introduction
   - Holonomy groups of spin connections and metrics
3. Holonomy groups of Riemannian and Lorentzian manifolds
   - Riemannian manifolds
   - Lorentzian manifolds
4. Globally hyperbolic Lorentzian manifolds with special holonomy and parallel spinors
   - globally hyperbolic manifolds
   - A construction

Spinors on curved spaces

Let $(M^{p,q}, g)$ be a pseudo-Riemannian spin manifold ($w_2(M) = 0$). Then on (M, g) there is a special complex vector bundle $S := Q \times_{Spin(p,q)} \Delta$ (spinor bundle) with a covariant derivative $\nabla^S : \Gamma(S) \longrightarrow \Gamma(T^*M \otimes S)$ (spin connection) and a hermitian inner product $< \cdot, \cdot >$. ($n = p + q \geq 3$)

One can multiply vectors and spinors

$$X \in TM,\ \varphi \in S \longmapsto X \cdot \varphi \in S \qquad \text{Clifford product}$$

such that the following rules hold

- $(X \cdot Y + Y \cdot X) \cdot \varphi = -2g(X, Y)\, \varphi$
- $< X \cdot \varphi, \psi > = (-1)^{p-1} < \varphi, X \cdot \psi >$
- $\nabla^S_X (Y \cdot \varphi) = (\nabla^g_X Y) \cdot \varphi + Y \cdot \nabla^S_X \varphi$
- $X(< \varphi, \psi >) = < \nabla^S_X \varphi, \psi > + < \varphi, \nabla^S_X \psi >$

Conformally invariant operators on spinors

The Clifford product $\mu : T^*M \otimes S \longrightarrow S$ gives a splitting of the bundle of 1-forms on M with values in the spinor bundle

$$T^*M \otimes S = Im\mu \oplus Ker\mu = S \oplus Tw$$

Hence, there are two differential operators of 1-order on spinor fields

$$D := pr_S \circ \nabla^S \quad : \quad \Gamma(S) \to \Gamma(T^*M \otimes S) \to \Gamma(S)$$ **Dirac operator**

$$P := pr_{Tw} \circ \nabla^S \quad : \quad \Gamma(S) \to \Gamma(T^*M \otimes S) \to \Gamma(Tw)$$ **Twistor operator**

Conformal covariance:

$$\begin{aligned} D(e^{2\sigma}\hat{g}) &= e^{-\frac{n+1}{2}\sigma} D(g)\, e^{\frac{n-1}{2}\sigma} \\ P(e^{2\sigma}\hat{g}) &= e^{-\frac{1}{2}\sigma} P(g)\, e^{-\frac{1}{2}\sigma} \end{aligned}$$

Twistor equation: $P\varphi = 0$ **(φ conformal Killing spinor)**

$$P\varphi = 0 \;\Leftrightarrow\; \nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$$

Overdet. PDE $\Rightarrow$ talk of A. Cap

1. Op. in a BGG talk of R. Gover

Conformal Killing spinors and special Lorentzian geometries

Question: For which Lorentzian spin manifolds there exist solutions of the conformal Killing spinor equation: $\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$ for all vector fields X

Remark: The name comes from the following fact:

$\varphi \in \Gamma(S) \Rightarrow V_\varphi \in \mathcal{X}(M)$: $\quad g(V_\varphi, X) := - < X \cdot \varphi, \varphi >$

If φ is a conformal Killing spinor, than V_φ is a time- or lightlike conformal vector field with the same zeros as φ.

Maximal number of solutions

If (M, g) is conformally flat and 1-connected, then the conformal Killing spinor equation has the maximal possible number of independent solutions, namly $2^{[\frac{n}{2}]+1}$.
Any metric with this number of independent solutions is conformally flat.
(cf. talk of A. Cap).

What else in the non-conformally flat case ??

Theorem: (F. Leitner 2004)

Let (M, g) be a Lorentzian manifold with ”generic” conformal Killing spinor, then (M, g) is locally conformal equivalent to one of the following spaces

- Product of $(\mathbb{R}, -dt^2)$ with a Ricci-flat Riemannian manifold with parallel spinors.
- Lorentzian Einstein-Sasaki manifold
- Lorentzian Einstein-Sasaki manifold $\times (N, h)$, where (N, h) is a Riemannian Einstein-Sasaki manifold, a 3-Sasaki-manifold, a nearly Kähler manifold or a Riemannian sphere
- Fefferman space (cf. talk of K. Hirachi)
- Brinkman space with parallel spinor

φ is ”generic” iff

* φ has no zeros
* V_φ does not change the causal type
* $V_\varphi^\flat$ has constant rank, where rank σ = $max\{k \mid \sigma \wedge (d\sigma)^k \neq 0\}$

All these special geometries are interesting. The first 4 types are quite well understood.

Aim: Understand the last class of geometries: Brinkmann spaces

Definition:

A Brinkmann space is a Lorentzian manifold with a parallel lightlike vector field.

Such a manifold has special holonomy !!!

Question: What is known about Lorentzian manifolds with special holonomy ???

Holonomy groups and parallel sections

E vector bundle over M with covariant derivative ∇, $x \in M$

$$Hol_x(E,\nabla) := \{\mathcal{P}_\gamma^\nabla : E_x \to E_x \text{ parallel transport along} \gamma \mid \gamma \text{ loop in } x\}$$

P G-principal bundle over M with principal bundle connection ω, $p \in P_x$

$$Hol_p(P,\omega) := \{g \in G \mid \exists \text{ loop } \gamma \text{ in } x \text{ such that } \gamma_p^*(1) = p \cdot g\}$$

Let $\rho : G \to GL(V)$ be a representation, $E := P \times_G V$ and $\nabla = \nabla^\omega$.
Fixing a $p \in P_x$ gives an isomorphism $E_x \simeq V$ such that

$$Hol_x(E,\nabla^\omega) = \rho(Hol_p(P,\omega))$$

Holonomy principle: There is a 1-1 correspondence between

$$\{\varphi \in \Gamma(E) \mid \nabla^\omega \varphi = 0\} \quad \text{and} \quad \{v \in V \mid \rho(Hol_p(P,\omega))v = v\}$$
$$= \{v \in V \mid \rho_*(\mathfrak{hol}_p(P,\omega))v = 0\} \text{ if } \pi_1(M) = 0$$

Holonomy groups of spin connections and metrics

Let $(M^{p,q}, g)$ be a spin manifold with the frame bundle P and spin structure (Q, f).
$\lambda : Spin(p,q) \longrightarrow SO(p,q)$ 2-fold covering.

$$TM := P \times_{SO(p,q)} \mathbb{R}^{p,q} \qquad p \in P_x \text{ frame in } x$$
$$S := Q \times_{Spin(p,q)} \Delta \qquad q \in Q_x \text{ spin frame in } x,\ f(q) = p$$

$$\begin{aligned} Hol_x(TM, \nabla^g) &= Hol_p(P, \omega^{LC}) \subset SO(p,q) \\ Hol_x(S, \nabla^S) &= \rho(Hol_q(Q, \omega^{LC})) \subset \rho(Spin(p,q)) \end{aligned}$$

Then

$$\begin{aligned} \lambda(Hol_q(Q, \omega^{LC})) &= Hol_x(TM, \nabla^g) \\ \mathfrak{hol}_q(Q, \omega^{LC}) &= (\lambda_*)^{-1} \mathfrak{hol}_x(TM, \nabla^g) \end{aligned}$$

Upshot: One can decide the existence of parallel spinors if one knows the holonomy group of (M, g). If (M, g) is simply connected, then

$$\{\varphi \in \Gamma(S) \mid \nabla^S \varphi = 0\} \equiv \{v \in \Delta \mid \rho_*(\lambda_*^{-1}(\mathfrak{hol}(TM, \nabla^g)))v = 0\}$$

Holonomy groups of Riemannian manifolds

(M^n, g) Riemannian manifold, complete, simply-connected.

DeRham Spitting Theorem: (G.DeRham 1952)

$(M, g) \simeq \mathbb{R}^k \times (M_1, g_1) \times \cdots \times (M_k, g_k)$ where (M_i, g_i) is irreducible

Berger's List: (M. Berger 1955)

Let (M^n, g) be an irreducible non-locally symmetric Riemannian manifold. Then the holonomy group $Hol(M, g)_0$ is (up to conjugation) one of the following once

$SO(n)$	generic type	0
$U(\frac{n}{2})$	Kähler	0
$SU(\frac{n}{2})$	Ricci-flat, Kähler	2
$Sp(\frac{n}{4})$	Hyperkähler	$\frac{n}{4} + 1$
$Sp(\frac{n}{4}) \cdot Sp(1)$	quaternionic Kähler	0
G_2	$n = 7,$ special parallel 3-form	1
$Spin(7)$	$n = 8,$ special parallel 4-form	1

Holonomy groups of symmetric spaces

Let (M, g) be a 1-connected symmetric space, $M = G/K$, where $G \subset Isom(M, g)$ is the transvection group of M and $K = G_x$ the stabilizer of a point $x \in M$. Then

1. $Hol_x(M, g) \simeq K$

2. The holonomy representation $Hol_x(M, g) \longrightarrow SO(T_xM, g_x)$ is given by the isotropy representation of K.

irreducible symmetric spaces are classified $\Rightarrow$ their holonomy groups are known

Holonomy groups of Lorentzian manifolds

(M^n, g) Lorentzian manifold, complete, simply-connected.

Wu Spitting Theorem: (H.Wu 1967)

$$(M, g) \ \simeq \ (N, h) \times (M_1, g_1) \times \cdots \times (M_k, g_k),$$

where (M_i, g_i) are flat or irreducible Riemannian manifolds and (N, h) is a Lorentzian manifold that is eigther

- flat
- irreducible or
- weakly irreducible and non-irreducible, t.m. the holonomy representation $\rho : Hol(N, h) \longrightarrow SO(T_xM, g_x)$ has no non-degenerate invariant subspace, but a degenerate invariant one.

Theorem 1 (Berger's list, Olmos/Di Scala'01, Boubel/Zeghib'03, Benoist/delaHarpe'04)

If the holonomy group $Hol(N,h)$ of a simply connected Lorentzian manifold acts **irreducible** than

$$Hol(N,h) = SO_0(1,n-1)$$

There is no special irreducible Lorentzian holonomy !!!

Let $Hol(N,h)$ act **weakly-irreducible and non-irreducible**.
If W is a degenerate invariant subspace, then $W \cap W^{\perp} = \mathbb{R}v_0$ for a lightlike vector v_0. Hence

$$Hol(N,h) \subset SO(1,n-1)_{\mathbb{R}v_0} = (\mathbb{R}^* \times SO(n-2)) \ltimes \mathbb{R}^{n-2}$$

Theorem 2 (Berard-Bergery/Ikemakhen '93, Galaev'04)

Let $\mathfrak{h} \subset \mathfrak{so}(1, n-1)_{\mathbb{R}v_0} = (\mathbb{R} \oplus \mathfrak{so}(n-2)) \ltimes \mathbb{R}^{n-2}$ be a weakly-irreducible sub-algebra and $\mathfrak{g} := proj_{\mathfrak{so}(n-2)}(\mathfrak{h}) = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{so}(n-2)$.

Then there are 4 cases

- $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{n-2}$
- $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^{n-2}$
- $\mathfrak{h} = (graph(\varphi) \oplus [\mathfrak{g}, \mathfrak{g}]) \ltimes \mathbb{R}^{n-2}$,

  where $\varphi : \mathfrak{z}(\mathfrak{g}) \longrightarrow \mathbb{R}$ is linear and surjective
- $\mathfrak{h} = ([\mathfrak{g}, \mathfrak{g}] \oplus graph(\psi)) \ltimes \mathbb{R}^{r}$,

  where $\mathbb{R}^{n-2} = \mathbb{R}^r \oplus \mathbb{R}^s$, $0 < r, s < n-2$
  $\mathfrak{g} \subset \mathfrak{so}(\mathbb{R}^r)$
  $\psi : \mathfrak{z}(\mathfrak{g}) \longrightarrow \mathbb{R}^s$ linear and surjecive

Theorem 3 (Th. Leistner 2003)

Let (N^n, h) be a simply-connected Lorentzian manifold with a weakly irreducible and non-irreducible acting holonomy group $Hol(N,h)$ and let $G := proj_{SO(n-2)} Hol(N,h) \subset SO(n-2)$. Then

- G is the product of Riemannian holonomy groups.

- (N,h) has parallel spinors if and only if

$$Hol(N,h) = G \ltimes \mathbb{R}^{n-2},$$

  where G is trivial or a product of $SU(k), Sp(l), G_2$ or $Spin(7)$.

Theorem 4 (A. Galaev 2005)

Any group appearing in Theorem 2 and Theorem 3 is in fact the holonomy group of a Lorentzian manifold.

A. Galaev constructed local analytic metrics for all types
(the coupled types where unknown before):

$$N = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$$

$$h_{(t,s,x)} = 2dtds + f(t,s,x)ds^2 + 2\sum_{j=1}^{n_0} u^j(s,x)dx^j ds + \sum_{j=1}^{n-2}(dx^j)^2$$

- $f(t,s,x) = \ldots$ special form in the four cases, uses the coupling functions φ and ψ
- $u^j(s,x) = A^j_{\alpha ik} x^i x^k s^{\alpha-1}$ $\quad A^{\cdot}_{\cdots}$ comes from a basis of $\mathfrak{g} = proj_{\mathfrak{so}(n-2)}\mathfrak{hol}(N,h)$.

$\Longrightarrow$ **The classification of holonomy groups of simply-connected Lorentzian manifolds is finished**

Task: Describe global models for Lorentzian manifolds with special holonomy

- Lorentzian symmetric spaces are known: Space forms (Minkowski, AdS, dS) and Cahen-Wallach spaces (Cahen/Wallach 1970)
  (Approach to classification of weakly-irreducible symmetric spaces (non-semisimple transvection group) by I. Kath, M. Olbrich (2004))
- Lorentzian homogeneous spaces (open problem, Th. Neukirchner)
- Globally hyperbolic Lorentzian manifolds (H. Baum, O. Müller (2005))
- Complete Lorentzian manifolds ???
- ???

Question: Which of the special Lorentzian holonomy groups can be realized by globally hyperbolic Lorentzian manifolds ?

Globally hyperbolic Lorentzian manifolds with special holonomy

Definition:

A Lorentzian manifold (M, g) is called globally hyperbolic iff

- (M, g) is stongly causal (for example if there exists a continous function f on M which is strictly increasing along any future directed causal curve)
- $J^+(p) \cap J^-(q) \subset M$ is compact for all $p, q \in M$

$$J^\pm(p) := \{x \in M \mid \exists \gamma : p \to x \text{ causal}, \uparrow_+ (\downarrow_-)\}$$

Some special properties of globally hyperbolic manifolds

- Normally hyperbolic operators have an global and unique forward and backward fundamental solution
- Existence of Cauchy surfaces
- Maximal causal geodesics: $p, q \in M$, $p \leq q$. Then there exists a causal geodesic from p to q of maximal length.

A (very) partial answer:

Theorem 5 (Baum/Müller 2005)
Any Lorentzian holonomy group of the form

$$G \ltimes \mathbb{R}^{n-2} \subset SO(1, n-1)$$

where $G \subset SO(n-2)$ is trivial or the product of groups of the form $SU(k), Sp(l), G_2$ or $Spin(7)$ can be realized by a globally hyperbolic Lorentzian manifold (M^n, g).

The idea for the construction of such metric was inspired by a paper of Ch. Bär, P. Gauduchon, A. Moroianu (2004)

A special construction

Let (M, g_0) be a Riemannian spin manifold with a Codazzi tensor A

A symmetric (1,1)-tensor field with $(\nabla^{g_0}_X A)(Y) = (\nabla^{g_0}_Y A)(X)$.

A spinor field $\varphi \in \Gamma(S_M)$ is called A- Codazzi spinor if

$$\nabla^S_X \varphi = iA(X) \cdot \varphi \qquad \text{for all vector fields } X$$

Theorem: (Bär/Gauduchon/Moroianu'04, Baum/Müller'05)
Let (M, g_0) be a complete Riemannian spin manifold with an A-Codazzi spinor, then the Lorentzian cylinder

$$C := I \times M \ , \ g_C := -dt^2 + (1 - 2tA)^* g_0$$

is globally hyperbolic with special holonomy and a parallel spinor.

Question: Are there A-Codazzi spinors ???
How such manifolds look like for invertable A ?

The case of invertable Codazzi tensors A

Theorem: (Baum/Müller'05)
Let (M, g_0) be a complete Riemannian manifold with A-Codazzi spinor for an invertable Codazzi tensor A, and let all eigenvalues of A are uniformally bounded away from zero. Then

$$(M, g_0) \simeq (\mathbb{R} \times F\ ,\ (A^{-1})^*(ds^2 + e^{-4s} g_F))$$

where (F, h) is a complete Riemannian manifold with parallel spinors and A^{-1} is a Codazzi-tensor on the warped product $(\mathbb{R} \times F, ds^2 + e^{-4s} g_F)$.

And vice versa.

Theorem: (Baum/Müller'05)
Let (F, g_F) be a complete Riemannian manifold with parallel spinors, T a Codazzi tensor on (F, g_F) with eigenvalues bounded from below. T defines s-parameter families of appropriate Codazzi tensors B on the warped product $(\mathbb{R} \times F, ds^2 + e^{-4s} g_F)$. Let

$$C(F, B) := I \times \mathbb{R} \times F \ , \ g_C := -dt^2 + (B - 2t)^*(ds^2 + e^{-4s} g_F).$$

Then

- (C, g_C) is a globally hyperbolic Brinkman space.
- If (F, h) has a flat factor, then $C(F, B)$ is decomposable.
- If (F, h) is (locally) a product of irreducible factors, then $C(F, B)$ is weakly irreducible and

$$Hol^0_{(0,0,x)}(C, g_C) = (B^{-1} \circ Hol^0_x(F, g_F) \circ B) \ltimes \mathbb{R}^{dim F}$$