Projective-type differential invariants of curves, and their associated PDEs of KdV-type

Institute for Mathematics and its Applications

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Assume $u : J \subset \mathbb{R}^2 \to \mathbb{R}P^1$ is a solution of the KdV Schwarzian evolution

$$u_t = u_1 S(u) = u_3 - \frac{3}{2}\frac{u_2^2}{u_1}$$

where $u_k = \frac{\partial^k u}{\partial x^k}$ and where

$$S(u) = \frac{u_3}{u_1} - \frac{3}{2}\left(\frac{u_2}{u_1}\right)^2$$

is the Schwarzian derivative of u.

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Then

$$S(u)_t = S(u)_3 + 3S(u)_1 S(u)$$

is a solution of the KdV equation.

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Then, curvature and torsion of the flow satisfy an equation equivalent to the Nonlinear Schrödinger equation. If

$$\phi = \kappa e^{i \int \tau dx}$$

then

$$\phi_t = i\phi_2 + \frac{i}{2}|\phi|^2\phi$$

(Hasimoto, 72)

Assume $u : J \subset \mathbb{R}^2 \to \mathbb{R}\mathrm{P}^1$ is a solution of the PSL(2)-invariant evolution

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where h depends on $S(u), S(u)_1, \ldots$

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then

$$S(u)_t = (D^3 + 2S(u)D + S(u)_1)h$$

where D is $\frac{d}{dx}$. The operator $D^3 + 2S(u)D + S(u)_1$ defines the second Hamiltonian structure for KdV.

Assume $u : J \subset \mathbb{R}^3$ is a solution of the Euclidean invariant evolution

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$$P = \begin{pmatrix} -(\tau D + D\tau) & D^2 \frac{1}{\kappa} D - \frac{\tau^2}{\kappa} D + D\kappa \\ D \frac{1}{\kappa} D^2 - D \frac{\tau^2}{\kappa} + \kappa D & D\left(\frac{\tau}{\kappa^2} D + D \frac{\tau}{\kappa^2}\right) D + \tau D + D\tau \end{pmatrix}$$

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Group-base definition

(Fels and Olver, after Cartan, Green, Griffiths and others)

A (left invariant) moving frame of order k is an equivariant map

$$\rho : J^{(k)}(\mathbb{R}, M) \to G$$

with respect to the prolonged action of G on $J^{(k)}(\mathbb{R}, M)$ and the (left) action of G on itself.

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Theorem 1 *Let* $\phi_g : G/H \to G/H$ *be given by,* $\phi_g(u) = g \cdot u$. *Let* ρ *be a group based moving frame with* $\phi_\rho(o) = \rho \cdot o = u$, *where* $o = [H] \in G/H$ *represents the class of* H. *Consider* $d\phi_\rho(o)$. *We can identify* $d\phi_\rho(o)$ *with an element of* $GL(n)$, *where* $n = \dim M$.

The matrix $d\phi_\rho(o)$ *contains in its columns a classical moving frame.*

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The traditional road to finding differential invariants has been to find the Serret-Frenet equations, an equation whose solution is given by a classical moving frame.

Classical Frenet equations do NOT, in general, provide a complete generating set of differential invariants

Definition 1 *Consider* $K dx$ *to be the horizontal component of the pullback of the (left invariant) Maurer-Cartan form of the group* G *via* ρ. *That is,*

$$K = \rho^{-1} (\rho)_x \in \mathfrak{g}.$$

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Theorem 2 *(From Olver and Fels) Let* ρ *be a (left or right) moving frame of minimal order for a curve* u*. Then, the coefficients of the (left or right) Serre-Frenet equations for* ρ *generate a basis for the set of differential invariants of the curve. That is, any other differential invariant is a function of the entries of* K *and their derivatives with respect to* x*.*

If G is semisimple and G/H is flat the Lie algebra $\mathfrak{g}$ splits

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$$

and, locally $G = G_1 \cdot G_0 \cdot G_{-1}$ with $H = G_0 \cdot G_1$. The subgroup G_0 is called the linear isotropy subgroup and it is the component of G that acts linearly on G/H.

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Choose a minimal order moving frame ρ with $\rho \cdot u = o = [H] \in G/H$ and let $K = \rho^{-1}\rho_x$ be its Serret-Frenet equations. We split K according to the gradation.

$$K = K_1 + K_0 + K_{-1}.$$

One can prove that the term K_{-1} is constant.

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Definition 2 *We call differential invariants of projective type those invariants that appear in* K_1*.*

Example 1

Consider $\mathbb{R}P^1 \cong PSL(2)/H$, where $PSL(2)$ acts on $u \in \mathbb{R}P^1$ via fractional transformations

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If $A \in PSL(n+1)$, it can be locally factored as

$$A = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

A (left invariant) moving frame for the projective line is given by

$$\rho = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1^{\frac{1}{2}} & 0 \\ 0 & u_1^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \left(u_1^{-\frac{1}{2}}\right)_x & 1 \end{pmatrix}$$

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$$\mathrm{Ad}\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right)\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & \beta^{-1}\alpha u \\ 0 & 1 \end{pmatrix}$$

and, therefore, a classical moving frame for the projective line is $\beta^{-1}\alpha = u_1$

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A classical moving frame would produce NO differential invariant.

Let $u : J \subset \mathbb{R}^2 \to G/H$, G semisimple, G/H flat.

Theorem 3 *(85% in progress) The geometric Poisson bracket associated to curves on* G/H *can be further restricted to the submanifold of projective differential invariants. The resulting Poisson structures and the behavior of the invariants of projective type under geometric flows are either Projective, Conformal or Lagrangian like, always of KdV-type.*

Lagrangian behaviour

Let $G = Sp(2n)$. If $g \in G$ then, locally

$$g = \begin{pmatrix} I & u \\ 0 & I \end{pmatrix} \begin{pmatrix} \Theta & 0 \\ 0 & \Theta^{-T} \end{pmatrix} \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}$$

with u and S symmetric and $\Theta \in GL(n)$.

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with u and S symmetric and $\Theta \in GL(n)$.

The manifold G/H is called the Lagrangian Grassmanian.

Theorem 4 *There exists a minimal order moving frame* ρ *along a curve of Lagrangian planes such that its Serret-Frenet equations are given by*

$$K = \rho^{-1}\rho_x = \begin{pmatrix} K_0 & I \\ K_1 & K_0 \end{pmatrix}$$

where K_0 *is skew-symmetric and contains all differential invariants of order 4, and where* $K_1 = -\frac{1}{2}\mathcal{S}_d$ *with* $\mathcal{S}_d$ *diagonal and containing along the diagonal the eigenvalues of the Lagrangian Schwarzian derivative (Ovsienko 94)*

$$\mathcal{S}(u) = u_1^{-1/2}\left(u_3 - \frac{3}{2}u_2u_1^{-1}u_2\right)u_1^{-1/2}.$$

Theorem 5 *Assume* $u : J \subset \mathbb{R}^2 \to Sp(2n)/H$ *is a flow solution of*

$$u_t = \Theta^T u_1^{1/2}\ h\ u_1^{1/2} \Theta$$

where $\Theta(x,t) \in O(n)$ *diagonalizes* $\mathcal{S}(u)$ *and where* h *is an invariant symmetric matrix.*

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Assume we choose initial conditions for which $K_0 = 0$*, then* $\mathcal{S}_d$ *satisfies the equation*

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If $h = \mathcal{S}_d$, *then* $\mathcal{S}_d$ *is the solution of a decoupled system of KdV equations*

$$(\mathcal{S}_d)_t = (\mathcal{S}_d)_{xxx} + 3\mathcal{S}_d(\mathcal{S}_d)_x.$$

Conformal behaviour

Consider $G = O(n+1,1)$. If $g \in G$ then

$$g = \begin{pmatrix} 1 & u^T & \frac{1}{2}u^T u \\ 0 & I & u \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ v & I & 0 \\ \frac{1}{2}v^T v & v^T & 1 \end{pmatrix}$$

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The manifold $O(n+1,1)/H$ is the Möbius sphere, the local model for flat conformal manifolds.

Theorem 6 *There exists a minimal order (left) moving frame* ρ *along a curve in the Möbius sphere such that its Serret-Frenet equations are given by*

$$K = \begin{pmatrix} 0 & e_1^T & 0 \\ k_1 e_1 + k_2 e_2 & K_0 & e_1 \\ 0 & k_1 e_1^T + k_2 e_2^T & 0 \end{pmatrix}$$

where k_1 *and* k_2 *are third order conformal invariants of projective type and where* K_0 *are fourth order and higher.*

Theorem 7 *Assume* $u : J \subset \mathbb{R}^2 \to O(n+1,1)/H$ *is a solution of*

$$u_t = h_1 T + h_2 N$$

where T *and* N *are conformal tangent and normal. Then the flow preserves* K_0.

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$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}D^3 + k_1 D + D k_1 & k_2 D + D k_2 \\ k_2 D + D k_2 & \frac{1}{2}D^3 - k_1 D - D k_1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

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In particular, if $h_1 = k_1$ *and* $h_2 = k_2$, *then* k_1 *and* k_2 *are solutions of a complexly coupled system of KdV equations*

Projective behaviour

Let $G = \mathrm{PSL}(n+1)$. If $g \in G$ then, locally

$$g = \begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ v^T & 1 \end{pmatrix}.$$

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The manifold $\mathrm{PSL}(n+1)/H$ can be identified with projective $\mathbb{R}\mathrm{P}^n$.

Theorem 8 *(Wilczynski 1906) There exists a moving frame* ρ *along a curve in* $\mathbb{R}P^n$ *such that its Serret-Frenet equations are given by*

$$K = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ k_1 & k_2 & \dots & k_n & 0 \end{pmatrix}$$

where k_i*,* $i = 1, \dots, n$ *are the Wilczynski projective invariants.*

Theorem 9 *Assume* $u : J \subset \mathbb{R}^2 \rightarrow PSL(n)/H$ *is a solution of*

$$u_t = h_1T_1 + h_2T_2 + \ldots + h_nT_n$$

where T_i *form a projective classical moving frame.*

Theorem 9 *Assume* $u : J \subset \mathbb{R}^2 \to PSL(n)/H$ *is a solution of*

$$u_t = h_1 T_1 + h_2 T_2 + \ldots + h_n T_n$$

where T_i *form a projective classical moving frame. Then, Wilczynski invariants satisfy an equation of the form*

$$\mathbf{k}_t = P\mathbf{h}$$

where $\mathbf{k} = (k_1, \ldots, k_n)^T$, $\mathbf{h} = (h_1, \ldots, h_n)^T$*and where* P *is the Poisson tensor defining the Adler-Gel'fand-Dikii Hamiltonian structure (Adler, Gel'fand, Dikii, Drinfel'd and Sokolov).*

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In particular, if $\mathbf{h} = \mathbf{k}$ *then* $\mathbf{k}$ *satisfies a generalized KdV system of equations.*

Both Poisson structures on $C^\infty(S^1, \mathfrak{g}^*)$ reduce to the submanifold of differential invariants to produce biHamiltonian systems.