

Outline

Geometry of regular  
(free) separation

Constrained (non-regular)  
separation

Examples

- fixed-energy separation
- reduced separation for  
the Schrödinger equation
- constrained R-sep for  
the Helmholtz equation

# A geometric approach to non-regular separation

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## Points of view on separation of variables (SoV) for a PDE

1. reduction of the PDE to a set of ODEs involving a single unknown: separated equations (Jacobi)
2. search of (families of) solutions of a special form e.g., additive SoV, multiplicative SoV etc.

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- ▶ For Hamilton-Jacobi equation 2. and 1. are equivalent, when the separated solution is complete.
  - ▶ Conditions for additive SoV for a Hamiltonian in a given coordinate system were found by Levi-Civita (1904).

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(free) separation

Constrained (non-regular)  
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Following their approach we reformulate their conditions in a more geometrical way, allowing us to give a 'positive' definition of the non-regular separation.

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separation

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# Geometry of regular separation

On a  $n$ -dimensional manifold  $Q$  we consider the  $l$ -th order PDE

$$H(q^i, u, u_j, u_{i_1 i_2}, \dots, u_{i_1 \dots i_l}) = h, \quad h \in \mathbb{R}$$

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If we look for separable solutions in the  $(q^i)$  i.e., solutions  $u = \sum_i S^{(i)}(q^i, c_A)$ , then the PDE can be written as

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(free) separation

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## Notation:

►  $u$  is the dependent variable (unknown function) and

$$u_i = \frac{\partial u}{\partial q^i}, \quad u_i^{(2)} = u_{ii} = \frac{\partial^2 u}{(\partial q^i)^2}, \quad \dots$$

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the Schrödinger equation

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- ▶  $Z$  is the  $(nl + 1)$ -dimensional space described by

$$(u, u_i, u_i^{(2)}, \dots, u_i^{(l)})$$

- ▶  $M = Q \times Z$  is the trivial bundle over  $Q$  with fiber  $Z$ .

## Conditions for SoV of $\mathcal{H} = h$ (Benenti, C., Rastelli)

The PDE (1) is separable in the  $(q^i)$  if and only if there exist  $n$  commuting symmetries of the PDE of the form

$$\mathcal{D}_i = \frac{\partial}{\partial q^i} + u_i \frac{\partial}{\partial u} + u_i^{(2)} \frac{\partial}{\partial u_i} + \dots + R_i \frac{\partial}{\partial u_i^{(l)}}, \quad (2)$$

where  $R_i(q^j, u, u_j, \dots, u_j^{(l)})$  are functions on  $M$ ,

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(free) separation

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$$\mathcal{D}_i \mathcal{H} = 0, \quad (3)$$

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### Outline

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(free) separation

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separation

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- ▶ condition (3) completely determines functions  $R_i$ ,
- ▶ condition (4) reduces to  $\mathcal{D}_i R_j = 0$  for  $i \neq j$  (integrability conditions of Kalnins and Miller).

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separation

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Let the vector fields  $\mathcal{D}_i$  be such that

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## Outline

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(free) separation

Constrained (non-regular)  
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the Schrödinger equation
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- ▶  $\Delta = \text{span}(\mathcal{D}_i)$  is an integrable distribution on  $M$ ; its  $n$ -dimensional integral manifolds are described by a *complete separated solution* of  $\mathcal{H} = h$ :

$$\begin{cases} u = S \\ u_i = S_i \\ u_{ij} = S_{ij} \\ \dots \end{cases} \quad \text{with } \partial_i S = S_i, \quad \partial_{ij}^2 S = \delta_{ij} S_{ii}, \quad \dots$$

## Outline

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(free) separation

Constrained (non-regular)  
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- ▶ free (regular) separated solutions depend on  $nl + 1$  parameters
- ▶ completeness means that for any  $P_0 = (q_0^i)$  there is a separated solution of  $\mathcal{H} = h$  for each choice of the value of  $u$  and its  $nl$  'separated' derivatives  $u_i, u_{ij}, \dots$  at  $P_0$ .

## Outline

Geometry of regular  
(free) separation

Constrained (non-regular)  
separation

## Examples

– fixed-energy separation

– reduced separation for  
the Schrödinger equation

– constrained R-sep for  
the Helmholtz equation

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Outline

Geometry of regular  
(free) separation

Constrained (non-regular)  
separation

Examples

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- reduced separation for  
the Schrödinger equation
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the Helmholtz equation

Kalnins and Miller call *non-regular separation* the case when

$$\mathcal{D}_i = \frac{\partial}{\partial q^i} + u_i \frac{\partial}{\partial u} + u_i^{(2)} \frac{\partial}{\partial u_i} + \dots + R_i \frac{\partial}{\partial u_i^{(l)}}, \text{ are such that}$$

$$\mathcal{D}_i \mathcal{H} = 0, \quad \text{but} \quad [\mathcal{D}_i, \mathcal{D}_j] \text{ are not identically zero.}$$

in this case “separable solutions still may exist, but they will depend on fewer than  $n/ + 1$  parameters.”

# An illustrative example (Kalnins and Miller)

Let us consider the  $2^{nd}$  order PDE in two variables  $(x_1, x_2)$

$$H = x_2 u_{11} + x_1 u_{22} + u_1 + u_2 = h.$$

In this case we have  $\dim Z = 2 \cdot 2 + 1 = 5$  and

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ii} \frac{\partial}{\partial u_i} + R_i \frac{\partial}{\partial u_{ii}},$$

$(i = 1, 2)$ . Being

$$\mathcal{D}_i H = u_{ii} + u_{jj} + R_i x_j, \quad i = 1, 2, j \neq i,$$

we get

$$R_i = -\frac{u_{ii} + u_{jj}}{x_j}, \quad i = 1, 2, j \neq i.$$

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We have

$$\mathcal{D}_i R_j = \frac{(u_{ii} + u_{jj})(x_i + x_j)}{x_i^2 x_j}, \quad i = 1, 2, j \neq i$$

which vanish for  $u_{11} + u_{22} = 0$  and under this constraint a non-regular separable solution involving 4 parameters  $(h, c_1, c_2, c_3)$  is

$$u = c_1 x_1^2 + c_2 x_1 - c_1 x_2^2 + (h - c_2)x_2 + c_3.$$

Outline

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However, both  $\mathcal{D}_i R_j$  vanish also for  $x_1 + x_2 = 0$ .

Outline

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Constrained (non-regular) separation

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However, both  $\mathcal{D}_i R_j$  vanish also for  $x_1 + x_2 = 0$ .

## Questions about non-regular SoV:

- ▶ how many parameters are involved?
- ▶ does it always occurs on the sets where  $\mathcal{D}_i$  commute?
- ▶ which is the geometrical interpretation?

Outline

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Examples

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# Constrained separation

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Let  $S$  be a  $r$  codimensional submanifold of  $M$  locally described by

$$f_\alpha = 0, \quad (\alpha = 1, \dots, r)$$

## Definition

The PDE  $\mathcal{H} = h$  admits a *constrained separation* on  $S$  if

- (1)  $u = \sum_i S^{(i)}(q^i, c_A)$  is a solution of  $\mathcal{H} = h$
- (2)  $u$  depends on  $nl + 1 - r$  parameters  $(c_A)$  such that

$$\text{rank} \left[ \frac{\partial u}{\partial c_A} \left| \frac{\partial u_i}{\partial c_A} \right| \dots \left| \frac{\partial u_i^{(l)}}{\partial c_A} \right| \right] = nl + 1 - r$$

- (3)  $u$  and its derivatives satisfy  $f_\alpha(q^j, u, u_i \dots) = 0$  for all  $(c_A)$ .

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## Proposition

Equation (1) admits constrained separation on  $S$  in the coordinates  $(q^i)$  if and only if the vector fields  $\mathcal{D}_i$  are: symmetries of (1), tangent to  $S$ , and commuting on  $S$ :

$$\mathcal{D}_i \mathcal{H} = 0, \quad \mathcal{D}_i f_\alpha|_S = 0, \quad [\mathcal{D}_i, \mathcal{D}_j]|_S = 0$$

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(free) separation

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separation

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- ▶  $\Delta_S = \text{span}(\mathcal{D}_i)$  is an integrable  $n$ -dimensional distribution on  $S$  whose integral manifolds are described by a constrained complete separated solution of  $\mathcal{H} = h$ .

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- ▶  $\Delta_S = \text{span}(\mathcal{D}_i)$  is an integrable  $n$ -dimensional distribution on  $S$  whose integral manifolds are described by a constrained complete separated solution of  $\mathcal{H} = h$ .
- ▶ constrained separated solutions depend on  $nl + 1 - r$  parameters
- ▶ completeness means that for any  $P_0 = (q_0^i)$  there is a separated solution of  $\mathcal{H} = h$  for each choice of the value of  $u$  and its  $nl$  separated derivatives at  $P_0$  satisfying  $f_\alpha = 0$  (i.e., the initial condition belongs to  $S$ ).

# An illustrative example (Kalnins and Miller)

Let us consider the  $2^{nd}$  order PDE in two variables  $(x_1, x_2)$

$$H = x_2 u_{11} + x_1 u_{22} + u_1 + u_2 = h.$$

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$$\text{Ex: } H = x_2 u_{11} + x_1 u_{22} + u_1 + u_2 = h$$

$$\mathcal{D}_i R_j = \frac{(u_{ii} + u_{jj})(x_i + x_j)}{x_i^2 x_j}, \quad i = 1, 2, j \neq i.$$

For  $u_{11} + u_{22} = 0$ , there is a non-regular separable solution involving 4 parameters  $(h, c_1, c_2, c_3)$

$$u = c_1 x_1^2 + c_2 x_1 - c_1 x_2^2 + (h - c_2)x_2 + c_3.$$

However, both  $\mathcal{D}_i R_j$  also vanish for  $x_1 + x_2 = 0$ .

The difference between the two cases is that  $\mathcal{D}_1, \mathcal{D}_2$  are tangent to the 1-codim submanifold

$$S = \{u_{11} + u_{22} = 0\}$$

$(\mathcal{D}_i(u_{11} + u_{22}))|_S = [(u_{11} + u_{22})/x_i]|_S = 0$ ) but not to

$$S_1 = \{x_1 + x_2 = 0\}.$$

Thus, on  $S$  there is a constrained separation, but not on  $S_1$ .

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Constrained (non-regular)  
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## Fixed-energy separation (Kalnins, Miller; B., C., R.)

Let us consider the separation of the single PDE  $\mathcal{H} = E$  ( $E$  fixed): it can be considered as a constrained separation on the hypersurface  $S = \{\mathcal{H} = E\}$ .

It occurs iff the  $\mathcal{D}_i$  commute on the single level set  $\mathcal{H} = E$ , i.e., the functions  $\mathcal{D}_i R_j$  are of the kind

$$\mathcal{D}_i R_j = (\mathcal{H} - E)\mu_{ij}, \quad \mu_{ij} \in \mathcal{F}(M),$$

the additional condition that  $\mathcal{D}_i$  are tangent to  $S$  is automatically satisfied: they are tangent to each submanifold  $\mathcal{H} = h$ ,  $h \in \mathbb{R}$ .

# Reduced separation for the Schrödinger equation

The multiplicative separation of the stationary Schrödinger eq.

$$\Delta\psi + (E - V)\psi = 0$$

i.e., search of complete solutions of the form  $\psi = \prod_i \psi_i(q^i, c_A)$ ,  
corresponds via  $u = \ln \psi$  to the additive SOV of the PDE

$$\mathcal{H} = g^{ii} u_{;i} + g^{ij} u_i u_j - \Gamma^i u_i - V = -E \quad (\Gamma^i = g^{hk} \Gamma_{hk}^i)$$

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Constrained (non-regular)  
separation

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# Reduced separation for the Schrödinger equation

The multiplicative separation of the stationary Schrödinger eq.

$$\Delta\psi + (E - V)\psi = 0$$

i.e., search of complete solutions of the form  $\psi = \prod_i \psi_i(q^i, c_A)$ ,  
corresponds via  $u = \ln \psi$  to the additive SOV of the PDE

$$\mathcal{H} = g^{ii} u_{ii} + g^{ij} u_i u_j - \Gamma^i u_i - V = -E \quad (\Gamma^i = g^{hk} \Gamma_{hk}^i)$$

## Outline

Geometry of regular  
(free) separation

Constrained (non-regular)  
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## Examples

- fixed-energy separation
- reduced separation for  
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## Definition (Benenti, C, Rastelli)

The separation is called reduced when  $r \leq n$  of the separated  
factors  $\psi_\alpha$  are of the form

$$\psi_\alpha = \exp(k_\alpha q^\alpha), \quad k_\alpha \in \mathbb{R}.$$

Thus, the coordinates are splitted into two kinds:  $(q^i) = (q^a, q^\alpha)$   
( $a = 1, \dots, m, \alpha = m + 1, \dots, n$ ).

# Conditions for reduced separation (KM, BCR)

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Reduced separation occurs (up to equivalence classes of separable coordinates) if and only if

1. the  $q^\alpha$  are ignorable coordinates
2. the  $q^a$  are pairwise orthogonal ( $g^{ab} = 0 \forall a \neq b = 1, \dots, m$ )
3. the metric is in standard form  $g^{ij} = \begin{pmatrix} \delta_b^a g^{ab} & 0 \\ 0 & g^{\alpha\beta} \end{pmatrix}$
4. the corresponding HJ equation is separable
5. the contracted Christoffel symbols satisfy  $\partial_a \Gamma_b = 0, \forall a \neq b$  i.e., the Ricci tensor is in standard form (reduced Robertson condition).

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## Proposition

The reduced separation of the Schrödinger equation is the constrained separation of

$$\mathcal{H} = g^{ii} u_{ii} + g^{jj} u_i u_j - \Gamma^i u_i - V = -E$$

on the submanifold  $S$  defined by the  $r$  equations  $u_{\alpha\alpha} = 0$ .

# Constrained $R$ -separation

Prus and Sym show two examples of non-regular  $R$ -separation of the Helmholtz equation on  $\mathbb{R}^3$

$$\Delta_3 \psi + E\psi = 0$$

occurring in coordinates which seem not related with separable coordinates for the geodesic HJ equation.

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separation

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## R-separation

It is the ansatz of assuming that there is a complete solution of the form

$$\psi = R(q^1, \dots, q^n) \prod_i \psi_i(q^i, c_A),$$

with  $\psi_i$  separated factors and  $R$  a nowhere 0 function on  $Q$ .

It generalises the multiplicative separation where  $R = 1$ .

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separation

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Geometry of regular  
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Under the assumption that the separated coordinates are orthogonal, by the change of unknown

$$u = \ln(\prod_i \psi_i(q^i)) = \ln \psi - \ln R,$$

$R$ -separation corresponds to the SOV of the PDE

$$H = g^{ii}(u_{ii} + u_i^2) - g^{ii}(\Gamma_i - 2\partial_i \ln R)u_i + R^{-1}\Delta R = -E$$

where  $\Gamma_i = g^{hk}\Gamma_{hki}$  and  $E \in \mathbb{R}$  is the energy constant.

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The functions  $D_i R_j$  are

$$D_i R_j = (S_{ij}(\hat{\Gamma}^h)u_h + u_{ii}(\partial_j \ln g^{ii}\hat{\Gamma}^i - \partial_j \hat{\Gamma}^i) + u_{jj}(\partial_i \ln g^{jj}\hat{\Gamma}^j - \partial_i \hat{\Gamma}^j) + S_{ij}(g^{hh})(u_h^2 - u_{hh}) - S_{ij}(R^{-1}\Delta R))\frac{1}{g^{jj}}, \quad (i \neq j \text{ n.s.})$$

where

$$S_{ij}(f) = \partial_{ij}^2 f - \partial_i \ln g^{jj} \partial_j f - \partial_j \ln g^{ii} \partial_i f,$$
$$\hat{\Gamma}^i = g^{ii}(\Gamma_i - 2\partial_i \ln R)$$

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## Proposition

Free  $R$ -separable solutions (in particular for all values of  $E$ ) exist if and only if

1. the corresponding geodesic HJ equation is separable, i.e.,

$$S_{ij}(g^{hh}) = 0$$

2.  $R$  is (up to separated factors) a solution of

$$\partial_i \ln R = \frac{1}{2} \Gamma_i \quad (5)$$

3.  $\frac{\Delta R}{R} = g^{hh} f_h(q^h)$  for suitable functions  $f_h$ .

# An example of non-regular separation

Prus and Sym show that in the orthogonal coordinates  $(q^1, q^2, q^3)$  of  $\mathbb{R}^3$  such that

$$\begin{aligned}g^{11} &= b \frac{(a \cosh q^2 - c \cos q^1)^2}{(a \cosh q^2 - q^3)^2} \\g^{22} &= b \frac{(a \cosh q^2 - c \cos q^1)^2}{(c \cos q^1 - q^3)^2} \\g^{33} &= 1\end{aligned} \quad a < c \in \mathbb{R}, \quad b = (a^2 - c^2)^{-1}$$

there exist non-regular  $R$ -separated solutions of  $\Delta_3 \psi + E\psi = 0$  of the form

$$\psi = [(q^3 - c \cos q^1)(a \cosh q^2 - q^3)]^{-1/2} \psi_1(q^1) \psi_2(q^2) \psi_3(q^3, E)$$

where  $\psi_i$  are solutions of

$$\psi_1'' + \psi_1/4 = 0, \quad \psi_2'' - \psi_2/4 = 0, \quad \psi_3'' + E\psi_3 = 0.$$

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(free) separation

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- ▶ the  $(q^i)$  do not allow SoV of the HJ equation ( $S_{ij}(g^{hh}) \neq 0$ );
- ▶ however, they satisfy requirements 2 and 3 for regular (free)  $R$ -separation

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- ▶ the  $(q^i)$  do not allow SoV of the HJ equation ( $S_{ij}(g^{hh}) \neq 0$ );
- ▶ however, they satisfy requirements 2 and 3 for regular (free)  $R$ -separation
- ▶ if  $R$  satisfies (5), then  $\hat{\Gamma}^i = g^{ii}(\Gamma_i - 2\partial_i \ln R) = 0$  and  $D_i R_j$  reduce to

$$D_i R_j = (S_{ij}(g^{hh})(u_h^2 - u_{hh}) - S_{ij}(R^{-1}\Delta R)) \frac{1}{g^{jj}}, \quad (i \neq j \text{ n.s.})$$

- ▶ system (5) is integrable if and only if

$$S_{ij}(g^{ii})/g^{ii} = S_{ij}(g^{jj})/g^{jj}$$

(a weaker condition than 1.) i.e., if and only if

$$\partial_{ij}^2 \log(g^{ii}/g^{jj}) = 0$$

# It's constrained separation!

A geometric approach to  
non-regular separation

C. Chanu

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## Proposition

If  $(q^i)$  are orthogonal coordinates on  $Q$  such that

$$\partial_{ij}^2 \log(g^{ii}/g^{jj}) = 0$$

$$\frac{\Delta R}{R} = g^{hh} f_h$$

$$S_{ab}(g^{cc}) = 0 \quad a, b, c = r+1, \dots, n, \quad a \neq b$$

$$\partial_\alpha g^{cc} = 0 \quad \alpha = 1, \dots, r$$

then we have constrained  $R$ -separation of  $\Delta\psi + E\psi = 0$  on the submanifold  $S$  defined by the  $r$  equations

$$u_{\alpha\alpha} - u_\alpha^2 + f_\alpha = 0 \quad (\Leftrightarrow \psi_\alpha'' + f_\alpha \psi_\alpha = 0) \quad \alpha = 1, \dots, r$$

where  $R$  is a solution of  $\partial_i \ln R = \frac{1}{2} \Gamma_i$ .

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In the example of Prus and Sym we have that the metric splits in a 1-dimensional trivially separable metric ( $g^{33} = 1$ ) while the other  $r = 2$  components of the metric satisfy

$$\partial_{12} \log(g^{11}/g^{22}) = 0.$$

Moreover, we have  $f_1 = -f_2 = 1/4, f_3 = 0$ .

## Proposition

If an orthogonal coordinate system  $(q^i)$  on  $\mathbb{R}^3$  allows constrained  $R$ -separation of Helmholtz equation with  $r = 2$ ,

$$(q^\alpha) = (q^1, q^2), \quad (q^a) = q^3,$$

then the components of the metric tensor are (up to rescaling)

$$g^{11} = \frac{(F_1 H_2 - H_1 F_2)^2}{(F_2 q^3 + H_2)^2}$$

$$g^{22} = \frac{(F_1 H_2 - H_1 F_2)^2}{(F_1 q^3 + H_1)^2} \quad \text{with } F_i = F_i(q^i), \quad H_i = H_i(q^i).$$

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$$g^{33} = 1.$$

- ▶ It is a necessary condition, other conditions on functions  $F_i$ ,  $H_i$  must be satisfied (as instance  $R_{1212} = 0$ ).
- ▶ The example of Prus and Sym occurs (up to rescaling) for  $F_1 = F_2 = -1$ ,  $H_1 = c \cos q^1$   $H_2 = a \cosh q^2$ .

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