

# The rigidity problem for Carnot groups

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The observations in this talk come from a paper in preparation by A. Čap, M. Cowling, F. De Mari, M. Eastwood and R. McCallum about the Heisenberg group and the flag manifold, and more general papers by Cowling, De Mari, A. Korányi and H.M. Reimann, one published [?] and one in preparation, as well as papers by McCallum (in preparation) and B. Warhurst [?].

A Carnot group  $N$  is a connected, simply connected nilpotent Lie group whose Lie algebra  $\mathfrak{n}$  is stratified, that is

$$\mathfrak{n} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \cdots \oplus \mathfrak{n}^s,$$

where  $[\mathfrak{n}^1, \mathfrak{n}^j] = \mathfrak{n}^{j+1}$  when  $j = 1, 2, \dots, s$  (we suppose that  $\mathfrak{n}^s \neq \{0\}$  and  $\mathfrak{n}^{s+1} = \{0\}$ ), and carries an inner product for which the  $\mathfrak{n}^j$  are orthogonal. We may put additional structure on  $\mathfrak{n}$ . The left-invariant vector fields on  $N$  which corresponds to elements of  $\mathfrak{n}^1$  at the identity span a subspace  $HN_p$  of the tangent space  $TN_p$  of  $N$  at each point  $p$  in  $N$ , and the corresponding distribution  $HN$  is called the horizontal subbundle of the tangent bundle  $TN$ . The commentators of length at most  $s$  of sections of  $HN$  span the tangent space at each point.

A smooth curve is called horizontal if its tangent vectors are all horizontal; the inner product on  $\mathfrak{n}^1$  can be used to compute the length of these tangent vectors, and hence the length of a horizontal curve can be computed by integration. Thus Carnot groups have well defined distance functions — the distance between two points being the infimum of the lengths of the horizontal curves joining them. In general, it is hard to work with this distance function.

Carnot groups are important as models for subriemannian geometry. A (smooth) subriemannian manifold  $M$  is a (smooth) manifold with a (smooth) subbundle  $HM$  of the tangent bundle  $TM$ ; further, there is a metric on  $HM$ . If the distribution  $HM$  is involutive, then it induces a foliation of  $M$ , and the interesting geometric phenomena live in the leaves of the foliation. Thus one usually supposes that  $H^s M = TM$  for some  $s$ , where  $H^k M_p$  is the subspace of  $TM_p$  spanned by commutators of order at most  $k$  of sections of  $HM$  at  $p$ . Thus the tangent space  $TM_p$  to  $M$  at  $p$  is filtered. The associated graded space carries the structure of a stratified Lie algebra, and the metric on  $HM_p$  can easily be

extended so that the graded Lie algebra becomes the Lie algebra of a Carnot group, this group is a model for  $M$ , at least if all  $H^k M$  are subbundles.

By forgetting about metrics, Carnot groups can be used as models for manifolds with distributions. These appear naturally in differential equations (jet spaces), non-holonomic mechanics, and in diverse other applications including modelling vision. By putting additional structure on Carnot groups, they can be used to model other objects, such as  $CR$  manifolds.

Suppose that  $U$  and  $V$  are connected open subsets of a Carnot group  $N$ , and define

$$\text{Contact}(U, V) = \{f \in \text{Diffeo}(U, V) : df(HU_p) = HV_{f(p)} \text{ for all } p \text{ in } U\}$$

(where  $\text{Diffeo}(U, V)$  denotes the set of diffeomorphisms from  $U$  to a subset of  $V$ ). The group is said to be rigid if  $\text{Contact}(U, V)$  is finite-dimensional. If  $N$  has additional structure, one might look at spaces of maps preserving this additional structure and ask whether the space of structure preserving maps is finite-dimensional. Much fundamental work on this question has been done by N. Tanaka [?] and his collaborators Morimoto [?] and K. Yamaguchi [?].

Rigidity is a blessing and a curse. It makes life simpler, but it makes it harder to choose coordinates. Following on from fundamental work of P. Pansu [?] on quasiconformal maps on Carnot groups, much effort has been expended studying quasiconformal and weakly contact maps. But if all Carnot groups except some relatively commutative examples were rigid, then much of this effort would be wasted.

For smooth maps, much is known about rigidity. If  $N$  is the nilradical of a parabolic subgroup of a semisimple Lie group  $G$ , then, as shown by Yamaguchi [?],  $N$  is rigid (and  $\text{Contact}(U, V)$  is essentially a subset of  $G$ ) except in a limited number of cases. If  $N$  is the free nilpotent group  $N_{g,s}$  of step  $s$  on  $g$  generators, then the situation is similar:  $N_{2,1}$  is non-rigid,  $N_{2,2}$  and  $N_{g,1}$  (where  $g \geq 2$ ) are rigid and  $\text{Contact}(U, V)$  is a subset of a semisimple group (in this case, some non-affine fractional linear transformations are contact maps) and otherwise  $N_{g,s}$  is rigid and  $\text{Contact}(U, V)$  is a subset of the affine group of  $N_{g,s}$ , that is, the semidirect product  $N_{g,s} \rtimes \text{Aut}(N_{g,s})$ , where the normal factor  $N_{g,s}$  gives us translations and the other factor automorphisms. Pansu [?] showed that the contact mappings of generic step 2 Carnot groups are affine, and Reimann [?] classified the rigid groups of Heisenberg type.

Most of these examples suggest that most Carnot groups are rigid, but jet spaces  $J^k(\mathbb{R}^m, \mathbb{R}^n)$  have a natural Carnot group structure of step  $k$ , and are non-rigid [?]. It is also known that knowing the dimensions of the spaces  $\mathfrak{n}^i$  is not enough to decide whether  $N$  is rigid or not.

Pansu's work allows us to define non-smooth contact maps, as those which map rectifiable curves into rectifiable curves (rectifiable curves are a natural generalisation of horizontal curves). It is reasonable to ask also whether such maps are automatically smooth. For a smooth map  $f$ , the stratification of  $\mathfrak{n}$  gives the differential  $df$  a block structure, where the  $(i, j)$ th block corresponds to maps from  $\mathfrak{n}^i$  to  $\mathfrak{n}^j$ . If  $f$  is a smooth contact map, then the  $(1, j)$ th blocks are zero when  $j \geq 2$ , and then considering commutators shows that

the  $(i, j)$ th blocks are zero when  $j \geq i + 1$ , i.e.,  $df$  is block upper diagonal. The diagonal part of  $df$  is a strata preserving automorphism of  $n$ . For non-smooth  $f$ , the derivative  $Pf$  defined by Pansu is a strata preserving automorphism which corresponds to the diagonal part of  $df$  for a smooth map. It is reasonable to ask what rigidity results can be proved for non-smooth maps of Carnot groups. One result in this direction, due to L. Capogna and Cowling [?], is that 1-quasiconformal maps are smooth. It then follows from work of N. Tanaka [?] that they form a finite-dimensional group.

The following example is prototypical of the rigid situation. Let  $N$  be the Heisenberg group of upper diagonal  $3 \times 3$  unipotent matrices

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

We define the left-invariant vector fields  $X, Y$ , and  $Z$  by

$$X = \partial/\partial x \quad Y = \partial/\partial y + x\partial/\partial z \quad Z = \partial/\partial z.$$

Then  $[X, Y] = Z$ .

Consider  $\{f \in \text{Diffeo}(U, V) : df(X) = pX, df(Y) = qY\}$ , where  $p$  and  $q$  are arbitrary functions. I would describe these as multicontact maps, as the differential preserves several subbundles of  $TN$ , not just one. A vector field  $M$  on  $U$  is called multicontact if  $M$  generates a flow of multicontact maps. This boils down to the requirement that  $[M, X] = p'X$  and  $[M, Y] = q'Y$ , where  $p'$  and  $q'$  are functions. If we write  $M$  as  $aX + bY + cZ$ , where  $a, b$  and  $c$  are functions, then we get the equations

$$\begin{aligned} - (Xa)X - (Xb)Y - (Xc)Z - bZ &= p'X \\ - (Xa)X - (Yb)Y - (Yc)Z + aZ &= q'Y, \end{aligned}$$

whence

$$\begin{aligned} Xb &= 0 & Xc &= -b \\ Yb &= 0 & Yc &= a \end{aligned}$$

We see immediately that  $c$  determines  $a$  and  $b$ , and that

$$X^2c = Y^2c = 0.$$

These equations imply that  $c$  is a polynomial in  $X, Y$  and  $Z$ , and in fact that the Lie algebra of multicontact vector fields is isomorphic to  $\mathfrak{sl}(3, \mathbb{R})$ . It then follows that the corresponding group of smooth multicontact mappings is a finite extension of  $SL(3, \mathbb{R})$ . However, even if care is taken, this argument can only work for mappings that are at least twice differentiable.

Now we consider the non-smooth case. The integrated version of the multicontact equations is quite simple. We denote by  $\mathcal{L}$  and  $\mathcal{M}$  the sets of integral curves for  $X$  and  $Y$ , i.e.,

$$\mathcal{L} = \{l_{y,z} : s \mapsto \begin{pmatrix} 1 & s & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : y, z \in \mathbb{R}^2\}$$

$$\mathcal{M} = \{m_{x,z} : t \mapsto \begin{pmatrix} 1 & x & z + xt \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : x, z \in \mathbb{R}^2\}.$$

and for open subsets  $U$  and  $V$  of  $N$ , we define  $\mathcal{L}_U$  to be the set of all connected components of the sets  $l \cap U$  as  $l$  varies over  $\mathcal{L}$ , and define  $\mathcal{M}_U$  similarly. We may now ask what can be said about maps which send line segments  $l$  in  $\mathcal{L}_U$  into lines in  $\mathcal{L}$  and line segments  $m$  in  $\mathcal{M}_U$  into lines in  $\mathcal{M}$ .

To answer this question, we first discuss related questions in  $\mathbb{R}^2$ .

**Theorem 0.1** *Let  $U$  be a connected open subset of  $\mathbb{R}^2$ , and let  $\varphi$  be a map from  $U$  into  $\mathbb{R}^2$  whose image contains 3 non-collinear points, and which maps connected line segments contained in  $U$  into lines. Then*

- (a) *if  $U = \mathbb{R}^2$  then  $\varphi$  is an affine transformation, i.e., the composition of a translation with a linear map;*
- (b) *otherwise,  $\varphi$  is a projective transformation.*

We do not prove this theorem here, but note that (a) is a theorem of Darboux; (b) requires more work. In the special case where  $U$  and  $\varphi(U)$  are both the unit disc, then (b) amounts to the identification of the geodesic-preserving maps of the hyperbolic plane (in the Klein model). It is worth observing that for any field (or division ring)  $k$ , the maps  $\varphi$  which preserve lines in  $k^2$  are compositions of field (or ring) automorphisms with affine maps, and that (b) can be extended to topological fields and division rings, and some other rings. We say that points  $P_1, \dots, P_4$  in the Heisenberg group  $N$  are in general position if neither of the sets  $\cup_{i=1}^4 \{(y, z) \in \mathbb{R}^2 : p_i \in l_{y,z}\}$  and  $\cup_{i=1}^4 \{(x, z) \in \mathbb{R}^2 : P_i \in m_{x,z}\}$  is contained in a line.

**Theorem 0.2** *Let  $U$  be a connected open subset of  $N$ , and let  $\varphi$  be a map from  $U$  into  $N$  whose image contains 4 points in general position, and which maps connected line segments in  $\mathcal{L}_U$  into lines in  $\mathcal{L}$  and connected line segments in  $\mathcal{M}_U$  into lines in  $\mathcal{M}$ . Then*

- (a) *If  $U = N$ , then  $\varphi$  is an affine transformation, i.e., the composition of a translation with a dilation  $(x, y, z) \mapsto (ax, ty, stz)$ .*
- (b) *Otherwise,  $\varphi$  is a projective transformation.*

**Proof.** We prove (a) only. There is a plane at infinity which compactifies the  $\mathcal{L}$  lines, which we write as  $\{(\infty, y, z) : y, z \in \mathbb{R}^2\}$ , and a projection  $\pi$  of  $N$  onto this plane, given by  $\pi : (x, y, z) \mapsto (\infty, y, z)$ . Any map  $\varphi$  of  $N$  to  $N$  which maps  $\mathcal{L}$  lines into  $\mathcal{L}$  lines induces a map  $\dot{\varphi}$  of this plane into itself. Any non-vertical line in the plane at infinity is of the form  $\pi(m)$  for some  $\mathcal{M}$  line  $m$ , and  $\varphi(m)$  is also an  $\mathcal{M}$  line. It follows that  $\dot{\varphi}$  sends non-vertical lines into non-vertical lines — an easy argument then implies that  $\dot{\varphi}$  maps all lines into lines. It follows that  $\varphi$  is an affine map, by Darboux’s theorem. One can compactify the  $\mathcal{M}$  lines similarly and obtain another map  $\ddot{\varphi}$  of a plane which preserves lines. Unravelling things,  $\varphi$  must be affine.

A similar proof works for the case where  $U \subset N$ , except that in this case one needs to show that a map of an open set in the plane which maps connected line segments, whose slopes lie in a small but open range, are projective.  $\square$

It is standard that collinearity preserving mappings of the projective plane are projective, i.e., arise from the action of  $PGL(3, \mathbb{R})$  on the projective space  $\mathbb{P}^2(\mathbb{R})$ . This is usually done by composing the collinearity preserving map with a projective map to ensure that the line at infinity in the projective plane is preserved, then applying Darboux’s theorem. This argument works on arbitrary fields, except for complications caused by Galois groups. But, essentially, Darboux’s theorem is equivalent to “the fundamental theorem of projective geometry”.

Theorem 0.2 (where  $U = N$ ) has a projective analogue, which apparently goes back to E. Bertini (though I have not been able to verify this). The projective completion of  $N$  is the flag manifold  $\mathbb{F}^3(\mathbb{R})$  whose elements are full flags of subspaces of  $\mathbb{R}^3$ , i.e., ordered pairs  $(V_1, V_2)$  of subspaces of  $\mathbb{R}^3$  such that

$$\{0\} \subset V_1 \subset V_2 \subset \mathbb{R}^3$$

(all inclusions are proper, so  $\dim(V_1) = 1$  and  $\dim(V_2) = 2$ ). Suppose that  $\varphi : \mathbb{F}^3(\mathbb{R}) \rightarrow \mathbb{F}^3(\mathbb{R})$  is a bijection. We write  $\varphi(V_1, V_2)$  as  $(\varphi_1(V_1, V_2), \varphi_2(V_1, V_2))$ . If  $\varphi_1(V_1, V_2)$  depends only on  $V_1$ , then  $\varphi$  induces a map  $\dot{\varphi}$  of  $\mathbb{P}^2(\mathbb{R}^3)$ ; similarly if  $\varphi_2(V_1, V_2)$  depends only on  $V_2$ , then  $\varphi$  induces a map  $\ddot{\varphi}$  of the set of planes in  $\mathbb{R}^3$ . If both these hypotheses hold, then the map  $\dot{\varphi}$  of  $\mathbb{P}^2(\mathbb{R})$  preserves collinearity: indeed, three collinear points in  $\mathbb{P}^2(\mathbb{R})$  correspond to three one-dimensional subspaces  $V_1, V_1'$  and  $V_1''$ , of  $\mathbb{P}^2(\mathbb{R})$  all contained in a common plane  $V_2$ , and by hypotheses  $\varphi_1(V_1, V_2)$ ,  $\varphi_1(V_1', V_2)$  and  $\varphi_1(V_1'', V_2)$  are all subspaces of  $\varphi_2(V_1, V_2)$ . Conversely, any collinearity-preserving map of  $\mathbb{P}^2(\mathbb{R})$  induces a map of  $\mathbb{F}^3(\mathbb{R})$ . From the affine theorem above, we obtain a corresponding theorem for  $F_3(\mathbb{R})$ . Let us agree first that 4 points in  $\mathbb{F}^3(\mathbb{R})$  are in general position if their one dimensional subspaces are in general position (no three lie in a plane) and their two dimensional subspaces are too (no three meet in a line).

**Theorem 0.3** *Let  $U$  be a connected open subset of  $\mathbb{F}^3(\mathbb{R})$ , and suppose that  $\varphi$  is a map from  $U$  into  $\mathbb{F}^3(\mathbb{R})$  whose image contains four points in general position. Suppose also that  $\varphi_2(V_1, V_2)$  depends only on  $V_1$  and  $\varphi_1(V_1, V_2)$  depends only on  $V_2$ . Then  $\varphi$  is a projective*

transformation, i.e.,  $\varphi$  arises from the action of (a finite extension of)  $PGL(3, \mathbb{R})$  on  $\mathbb{F}^3(\mathbb{R})$ .

In the case where  $U$  is all  $\mathbb{F}^3(\mathbb{R})$ , then this is a special case of theorem of Tits [?] (also called the fundamental theorem of projective geometry). The point here is that  $\mathbb{F}^3(\mathbb{R})$  may be identified with the group  $G/P$ , where  $G = SL(3, \mathbb{R})$  and  $P$  is the subgroup of  $G$  of lower triangular matrices. Indeed,  $G$  acts on  $\mathbb{R}^3$  and  $P$  is the stabiliser of the flag  $(\mathbb{R}\mathbf{e}_3, \mathbb{R}\mathbf{e}_3 + \mathbb{R}\mathbf{e}_2)$ , where  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  is the standard basis of  $\mathbb{R}^3$ , and the map  $g \mapsto (\mathbb{R}g\mathbf{e}_3, \mathbb{R}g\mathbf{e}_3 + g\mathbf{e}_2)$  identifies  $G/P$  with  $\mathbb{F}^3(\mathbb{R})$ . Note that the stabilisers of the sets  $\{(V_1, V_2) \in \mathbb{F}^3(\mathbb{R}) : V_1 = V\}$  and  $\{(V_1, V_2) \in \mathbb{F}^3(\mathbb{R}) : V_2 = V\}$  are the parabolic subgroups  $P_1$  and  $P_2$ :

$$P_1 = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\} \quad \text{and} \quad P_2 = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}.$$

Maps  $\varphi$  of  $\mathbb{F}^3(\mathbb{R})$  into itself with the property that  $\varphi_1(V_2, V_2)$  depends only on  $V_1$  correspond to maps of  $G/P$  which pass to maps of  $G/P_1$ , i.e., maps that preserve the fibration of  $G/P$  induced by  $P_1$  (the fibres are the sets  $xP_1/P$ ), and maps  $\varphi$  such that  $\varphi_2(V_1, V_2)$  depends only on  $V_2$  correspond to maps of  $G/P$  which preserve the fibration of  $G/P$  induced by  $P_2$ .

Using this formulation, Tits extended the fundamental theorem of projective geometry to semisimple Lie groups (of rank at least two). At the affine level, these extensions include theorems such as a characterisation of maps of the plane preserving circles; a local version of this theorem was proved by Caratheodory [?]. At this time, R. McCallum has proved local versions of Tits' Theorem for all the classical groups.

Our original problem (or one of them) was to decide what kind of rigidity results might hold for non-smooth maps of Carnot groups (or open subsets thereof). Here is a conjecture which holds in many examples, by Tits theorem, and which also holds in the smooth case.

**Conjecture 0.4** *Suppose that  $\mathfrak{n}_1$  and  $\mathfrak{n}_2$  are subalgebras of a stratified algebra  $\mathfrak{n}$ , such that  $\mathfrak{n}^1 = \mathfrak{n}_1^1 \oplus \mathfrak{n}_2^1$ , and for all  $X$  in  $\mathfrak{n}_1^1$ , there exists  $Y$  in  $\mathfrak{n}_2^1$  such that  $[X, Y] \neq 0$ , and vice versa. Let  $U$  be an open subset of the corresponding group  $N$ , and let  $\varphi : U \rightarrow N$  be a map which maps connected subsets of cosets of  $N_i$  into cosets of  $N_i$  when  $i = 1$  and  $2$ . Then  $\varphi$  is automatically smooth, and the set of these maps is finite-dimensional.*