Exterior differential systems for ordinary differential equations

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A. Naive language

Equations: $f(x, y, y', \dots, y^{(n)}) = 0$

Transformations:

- $\bar{x} = A(x, y)$, $\bar{y} = B(x, y)$ (point transformations);
- $\bar{x} = A(x, y, y')$, $\bar{y} = B(x, y, y')$, $\bar{y}' = C(x, y, y')$, where $d\bar{y} - \bar{y}'d\bar{x} = \lambda(dy - y'dx)$ (contact transformations).

Changes of variables are Lie pseudogroups.

B. Geometric language

Consider the jet space $J^n(\mathbb{R}, R)$ of smooth functions $f\colon \mathbb{R} \to \mathbb{R}$. More generally, we can consider the jet space $J^n(\mathbb{R}^2)$ of non-parametrized curves (=one-dimensional submanifolds) on the plane.

Local coordinates $(x, y_0, \dots, y_n)$ on $J^n(\mathbb{R}, \mathbb{R})$ are defined in such way that that the jet of the function $y(x)$ at point x_0 has coordinates $(x_0, y(x_0), y'(x_0), \dots, y^{(n)}(x_0))$.

Contact distribution:

$$C = \langle dy_0 - y_1\, dx, \dots, dy_{n-1} - y_n\, dx\rangle^{\perp}.$$

Contact transformations:

$$\phi\colon\ J^n(\mathbb{R}^2) \mapsto J^n(\mathbb{R}^2),\ \ \phi_* C = C.$$

Point transformations are those contact transformations that preserve the fibers of the projection $J^n(\mathbb{R}^2) \to \mathbb{R}^2$.

Ordinary differential equation is a hypersurface $\mathcal{E} \subset J^n(\mathbb{R}^2)$:

$$\mathcal{E} = \{f(x, y_0, \dots, y_n) = 0\}.$$

Contact (point) geometry of ordinary differential equations is the geometry of hypersurfaces in $J^n(\mathbb{R}^2)$ with respect to the pseudogroup of all contact (point) transformations

C. Equations as differential systems:

In the sequel we consider equations solved with respect to the highest derivative:

$$y^{(n+1)} = f(x, y, y', \dots, y^{(n)}).$$

Ordinary differential equation is a line bundle inside the contact distribution:

$$\begin{aligned} E &= \langle \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + \dots + y_n \frac{\partial}{\partial y_{n-1}} + f \frac{\partial}{\partial y_n} \\ &= \langle dy_0 - y_1\, dx, \dots, dy_{n-1} - y_n\, dx, dy_n - f\, dx \rangle^{\perp} \end{aligned}$$

Initial data in the equivalence problem

- pseudogroup $\mathcal{G}$;
- the class of equations $\{\mathcal{E}\}$ stable with respect to $\mathcal{G}$.

Two classes of equations

(1) Equations with sufficiently many symmetries. These equations can be solved via Lie integration method and its generalizations.

(2) *C-Class* equations: the equations whose invariants (with respect to the pseudogroup $\mathcal{G}$) are first integrals. If we have sufficiently many non-constant invariants, then the equation can be solved without any integration.

Example 1 (Sophus Lie, 1896)

Consider first order ODE:

$$y' = f(x, y)$$

with respect to one of the following pseudogroups:

(A) changes of dependent and independent variables:

$$(x, y) \mapsto (u(x), v(y));$$

(B) conformal transformations:

$$(x, y) \mapsto (u(x, y), v(x, y)), \qquad u_x = v_y, u_y = -v_x.$$

The equation can be considered as a line distribution

$$E = \langle \frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \rangle$$

on the plane.

For both pseudogroups we can construct a natural G-structure $P \subset \mathcal{F}(\mathbb{R}^2)$ on the plane with the structure group

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R}^* \right\}.$$

Namely:

(A) P consists of all frames $\{X_1, X_2\}$ such that $X_1 \in \langle \frac{\partial}{\partial x} \rangle$, $X_2 \in \langle \frac{\partial}{\partial y} \rangle$, $X_1 + X_2 \in E$;

(B) P consists of all conformal frames $\{X_1, X_2\}$ (i.e., $(X_1, X_2) = 0$, $(X_1, X_1) = (X_2, X_2)$) such that $X_1 \in E$.

Theorem. *There exists a unique torsion-free connection on P.*

Proof. Let $\theta_1 e_1 + \theta_2 e_2$ be the restriction of the canonical form to P. Let $\omega\colon TP \to \mathfrak{g}$ be an arbitrary connection form on P. Its torsion is equal to $\Theta_1 e_1 + \Theta_2 e_2$, where

$$\Theta_i = d\theta_i + \omega \wedge \theta_i = f_i \theta_1 \wedge \theta_2, \quad i = 1, 2.$$

Then the connection $\omega' = \omega - f_1\theta_1 + f_2\theta_2$ is torsion-free. In the same way we can show that any two torsion-free connections coincide. ☐

The curvature $\Omega = d\omega = I\theta_1 \wedge \theta_2$ defines the first invariant of the G-structure P. Two more invariants I_1, I_2 are determined by the equality:

$$dI = -2I\omega + I_1\theta_1 + I_2\theta_2.$$

The maximal dimension of the symmetry algebra $\mathrm{sym}(\mathcal{E})$ is 3. It is achieved if and only if I is constant, and, hence is 0.

Explicitly, we have:

(A) $I = (\ln f)_{xy}/f$,
  $I = 0$ if and only if $f(x, y) = p(x)q(y)$;
(B) $I = \Delta(\arctan f)/(1 + f^2)$,
  $I = 0$ if and only if $f = \tan u(x, y)$, $\Delta u = 0$.

The first integrals are:

(A) $\int (dy/q(y) - p(x)\,dx)$;
(B) $\int \exp(v)(\cos(u)\,dy - \sin(u)\,dx)$, where v is a harmonic conjugate to u.

It is possible to prove that the symmetry algebra can not be 2-dimensional. However, there exist equations with 1-dimensional symmetry algebra. This symmetry algebra is sufficient to solve the equation.

Example 2 (Elie Cartan, 1924)

Consider 2-nd order ODE $\mathcal{E}$:

$$y'' = f(x, y, y')$$

up to the pseudogroup of point transformations.

Consider the jet space $J^1(\mathbb{R}^2)$ with local coordinates $(x, y = y_0, z = y_1)$. Then we have the following vector distributions on $J^1(\mathbb{R}^2)$:

- contact distribution $C = \langle dy - z\,dx\rangle^{\perp}$;
- vertical distribution $V = \langle dx, dy\rangle^{\perp}$, which is preserved by all point (but not contact!) transformations;
- the line bundle $E = \langle dy - z\,dx, dz - f\,dx\rangle^{\perp}$ corresponding to the equation $\mathcal{E}$.

Pseudogroup of point transformations is defined as all (local) transformations preserving the flag $V \subset C$.

Invariants of the second-order ODE are invariants of the decomposition $C = V \oplus E$.

In general, we can consider any 3-dimensional manifold M with two line bundles E_1 and E_2 such that $E_1 \oplus E_2$ is non-degenerate. Then the exists a local diffeomorphism $\phi\colon M \to J^1(\mathbb{R}^2)$ such that $\phi_* E_1 = V$, $\phi_* E_2 = E$.

In terms of G-structures the decomposition $C = V \oplus E$ determines the G-structure with

$$G = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

In general G is not of finite type, and does not encode the additional information that the distribution C is non-degenerate.

Using Cartan's equivalence method it is possible to construct the absolute parallelizm on 8-dimensional bundle $P \to M$ and prove that the maximal dimension of symmetry algebra $\mathrm{sym}(\mathcal{E})$ is 8. This dimension is achieved only the trivial equation $y'' = 0$ or any equation equivalent to it.

However, this is not how Elie Cartan treated this case!

Cartan connections

Let us fix the model homogeneous space $M_0 = G/G_0$, and let $(\mathfrak{g}, \mathfrak{g}_0)$ be the corresponding pair of Lie algebras.

Definition. *Cartan connection* on a manifold M of the same dimension as M_0 is a principle G_0-bundle $\pi\colon P \to M$ and a 1-form $\omega\colon TP \to \mathfrak{g}$ such that:

(1) ω is an absolute parallelizm;
(2) $\omega(X^*) = X$ for all $X \in \mathfrak{g}_0$;
(3) $R_g^*\omega = (\operatorname{Ad} g^{-1})\omega$ for all $g \in G_0$.

The *curvature form* of a Cartan connection ω on P is a 2-form

$$\Omega = d\omega + 1/2[\omega, \omega]$$

on P with values in $\mathfrak{g}$. The curvature satisfies the following conditions:

(1) $R_g^*\Omega = (\operatorname{Ad} g^{-1})\Omega$ for all $g \in G_0$;
(2) $\Omega(X, Y) = 0$ if at least one of the tangent vectors X, Y is vertical;
(3) (structure equation) $d\Omega = [\Omega, \omega]$.

The *structure function* of the Cartan connection ω is a function $c\colon P \to \operatorname{Hom}(\wedge^2\mathfrak{g}/\mathfrak{g}_0, \mathfrak{g})$ defined by:

$$c_p\colon (X + \mathfrak{g}_0, Y + \mathfrak{g}_0) \mapsto \Omega_p\bigl(\omega_p^{-1}(X), \omega_p^{-1}(Y)\bigr)$$

The Cartan connection is said to be *flat*, if $\Omega = 0$ (or $c = 0$).

Cartan connection for second order ODE's

Model space is a flag manifold $M_0 = F_{1,2}(\mathbb{R}^3) = SL(3,\mathbb{R})/ST(3,\mathbb{R})$.

Any differential form ω with values in $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ can be uniquely written in the form

$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}, \quad \omega_{11} + \omega_{22} + \omega_{33} = 0,$$

where ω_{ij} are usual differential 1-forms.

Theorem. Let $C = V \oplus E$ be the structure corresponding to the second order ODE. Then there is a natural Cartan connection $\omega \colon TP \to \mathfrak{sl}(3,\mathbb{R})$ on a principle G_0-bundle $\pi \colon P \to J^1(\mathbb{R}^2)$ such that:

(1) for any section $s \colon J^1(\mathbb{R}^2) \to P$ we have

$$V = \langle s^*\omega_{21}, s^*\omega_{31}\rangle^{\perp},$$
$$E = \langle s^*\omega_{32}, s^*\omega_{31}\rangle^{\perp};$$

(2) the structure function c takes values in the subspace $W \subset \operatorname{Hom}(\wedge^2\mathfrak{g}/\mathfrak{g}_0, \mathfrak{g})$:

$$W = \left\{ e_{21} \wedge e_{31} \mapsto \left(\begin{smallmatrix} 0 & A & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), e_{31} \wedge e_{32} \mapsto \left(\begin{smallmatrix} 0 & 0 & C \\ 0 & 0 & D \\ 0 & 0 & 0 \end{smallmatrix}\right), e_{21} \wedge e_{32} \mapsto 0 \right\}.$$

In other words:

$$\Omega = \begin{pmatrix} 0 & A\omega_{21} & B\omega_{21} + C\omega_{32} \\ 0 & 0 & D\omega_{32} \\ 0 & 0 & 0 \end{pmatrix} \wedge \omega_{31}.$$

Outline of Cartan's proof

We can always assume that locally $P = \mathbb{R}^3 \times G$ is a trivial principal fiber bundle.

Let $s\colon \mathbb{R}^3 \to P$ be the trivial section and $\tilde{\omega} = s^*\omega$ be a form on $\mathbb{R}^3$ with values in $\mathfrak{g}$. Cartan shows that there exists a unique section $s\colon \mathbb{R}^3 \to P$ such that the form $\tilde{\omega} = s^*\omega$ satisfies the following conditions:

$$\begin{aligned} \tilde{\omega}_{21} &= dx, \\ \tilde{\omega}_{31} &= dy - x\,dz, \\ \tilde{\omega}_{32} &= dz - f\,dx, \\ (\tilde{\omega}_{22} - \tilde{\omega}_{11}) \wedge dx &= 0. \end{aligned}$$

Then applying these conditions to the structure function, Cartan computes that all other components of the form $\tilde{\omega}$ are uniquely determined and have the form:

$$\tilde{\omega}_{22} - \tilde{\omega}_{11} = \frac{\partial f}{\partial z} dx, \quad \tilde{\omega}_{33} - \tilde{\omega}_{11} = -\frac{1}{2}\frac{\partial^2 f}{\partial z^2}(dy - z\,dx),$$

$$\tilde{\omega}_{12} = \frac{\partial f}{\partial y} dx + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(dz - f\,dx) + \left(\frac{2}{3}\frac{\partial^2 f}{\partial y \partial z} - \frac{1}{6}\frac{d}{dx}\frac{\partial^2 f}{\partial z^2}\right)(dy - z\,dx),$$

$$\tilde{\omega}_{13} = \left(\frac{1}{3}\frac{\partial^2 f}{\partial y \partial z} - \frac{1}{6}\frac{d}{dx}\frac{\partial^2 f}{\partial z^2}\right) dx - \frac{1}{6} d\left(\frac{\partial^2 f}{\partial z^2}\right) + \mu(dy - z\,dx),$$

$$\tilde{\omega}_{23} = \frac{1}{6}\frac{\partial^3 f}{\partial z^3}(dy - z\,dx),$$

where

$$\mu = \frac{1}{6}\frac{\partial^3 f}{\partial y \partial z^2} - \frac{1}{6}\frac{\partial f}{\partial z} \cdot \frac{\partial^3 f}{\partial z^3} - \frac{1}{6}\frac{d}{dx}\frac{\partial^3 f}{\partial z^3}.$$

Fundamental invariants:

$$I_1 = s^*A = \frac{1}{6} f_{zzxx} - \frac{1}{6} f_z f_{zzx} - \frac{2}{3} f_{yzx} + \frac{2}{3} f_z f_{yz} + f_{yy} - \frac{1}{2} f_y f_{zz}$$

$$I_2 = s^*D = -\frac{1}{6} f_{zzzz}.$$

Other coefficients of the structure function:

$$s^*B = \frac{\partial \tilde{A}}{\partial z},$$

$$s^*C = -\frac{d}{dx}(\tilde{D}) - 2 f_z \tilde{A},$$

where

$$d/dx = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + f \frac{\partial}{\partial z}$$

is the operator of total derivative.

The following conditions are equivalent:

(1) the equation $y'' = f(x, y, y')$ is equivalent to the trivial equation $y'' = 0$;
(2) its symmetry algebra is 8-dimensional;
(3) $I_1 = I_2 = 0$.

C-class equations of second order

Theorem. *Let $I_1 = 0$. Then all invariants of the Cartan connection associated with the equation $\mathcal{E}$ are first integrals of this equation.*

Outline of the proof. If $I_1 = 0$ then the structure function vanishes identically on $\mathfrak{h}/\mathfrak{g}_0 \wedge \mathfrak{g}/\mathfrak{g}_0$, where

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \right\}$$

Let H be the corresponding subgroup of $G = SL(3, \mathbb{R})$. Then the connection ω can be considered as a Cartan connection with model G/H on the space $\mathcal{S}$ of all solutions of the equation $\mathcal{E}$. Hence, any invariant of the connection ω can be considered as a function on $\mathcal{S}$, i.e., is constant on each solution of $\mathcal{E}$.

Duality of second order equations There exists a local diffeomorphism ψ of the space $J^1(\mathbb{R}^2)$ that takes the pair of distributions (C, E) into the pair (C, V), so that the line bundle E becomes vertical, while the vertical line bundle V is transformed into some new line distribution E'. The second-order differential equation defined by the field E' is said to be *dual* to the equation corresponding to E.

In general there exist several contact transformations carrying E into V. Moreover, if ψ_1 and ψ_2 are two such transformations, then the mapping $\psi_1 \circ \psi_2^{-1}$ preserves the pair (C, V) and hence is a point transformation. Therefore the dual equation is well defined only up to point transformations.

The description of the dual equation can also be given in terms of the solutions of the initial second-order equation. They can be written in the form of a two-parameter family $F(x, y, u, v) = 0$, where u, v are parameters (say, $u = y(0)$ and $v = y'(0)$). If we now consider x and y as parameters, u as an independent and v as a dependent variable, we obtain a two-parameter family of curves in the plane, which coincides with the family of solutions of the dual second-order equation.

The duality transformation interchanges invariants I_1 and I_2. This, C-class equations are dual to the second order equations cubic with respect to y'.

Examples.

- $y'' = 0$ is self-dual.
- The equations $y'' = 1/y^3$ and $y'' = \frac{y'(1-(y')^2+((y')^2-1)^{3/2}}{x}$ are dual to each other. In particular, the second equation in this pair belongs to C-class.

Example 3 (Chern, 1939, Sato–Yoshikawa 1998)
Consider a third-order ordinary differential equation of the form

$$y''' = f(x, y, y', y'').$$

Let us fix the coordinate system (x, y_0, y_1, y_2) on $J^2(\mathbb{R}, \mathbb{R})$. In these coordinates, the completely integrable distributions E,V have the form

$$\begin{aligned} E &= \left\langle \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + y_2 \frac{\partial}{\partial y_1} + f(x, y_0, y_1, y_2) \frac{\partial}{\partial y_2} \right\rangle, \\ V &= \left\langle \frac{\partial}{\partial y_2} \right\rangle. \end{aligned}$$

Theorem (Tanaka, Sato–Yoshikawa). There exist a principal G_0-bundle $\pi\colon P \to \mathcal{E}$ and a Cartan connection ω of type G/G_0 on P, naturally associated with a given third-order equation, where $G = SP(4, \mathbb{R})$ and G_0 is a Borel subgroup in $SP(4, \mathbb{R})$. The corresponding pair of Lie algebras $(\mathfrak{g}, \mathfrak{g}_0)$ has the form

$$\mathfrak{g} = \mathfrak{sp}(4, \mathbb{R}) = \left\{ \begin{pmatrix} x_5 & x_7 & x_9 & x_{10} \\ x_1 & x_6 & x_8 & x_9 \\ x_3 & x_2 & -x_6 & x_9 \\ x_4 & x_3 & -x_1 & -x_5 \end{pmatrix} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} x_5 & x_7 & x_9 & x_{10} \\ 0 & x_6 & x_8 & x_9 \\ 0 & 0 & -x_6 & x_9 \\ 0 & 0 & 0 & -x_5 \end{pmatrix} \right\}.$$

Conditions on the curvature tensor

Basic ideas (due to Tanaka):

- the structure theory of graded Lie algebras:
$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0.$$

- interpreting the structure function of the Cartan connection as an element of cochain complex:
$$C^i(\mathfrak{m}, \mathfrak{g}) = \mathrm{Hom}(\wedge^i \mathfrak{m}, \mathfrak{g}) \qquad \partial\colon C^i(\mathfrak{m}, \mathfrak{g}) \to C^{i+1}(\mathfrak{m}, \mathfrak{g}).$$
The structure function takes values in $\mathrm{Hom}(\wedge^2 \mathfrak{g}/\mathfrak{g}_0, \mathfrak{g}) = \mathrm{Hom}(\wedge^2 \mathfrak{m}, \mathfrak{g})$;

- using Honge theory for Lie algebras:
$$(X, Y) = X^t Y, \qquad X, Y \in \mathfrak{sp}(4, \mathbb{R}),$$
$$\partial^* C^{i+1}(\mathfrak{m}, \mathfrak{g}) \to C^i(\mathfrak{m}, \mathfrak{g}).$$

The condition on curvature of the Cartan connection:
$$c\colon P \to C^2(\mathfrak{m}, \mathfrak{g}); \qquad \boxed{\partial^* c = 0}.$$

Fundamental invariants and C-class

Denote by $e_1, \dots, e_{10}$ the corresponding basis of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$. The curvature form of ω has the form $\Omega = \sum_{i=1}^{10} \Omega_i e_i$, where

$$\begin{aligned}
&\Omega_1 = \Omega_3 = \Omega_4 = \Omega_5 = 0;\\
&\Omega_2 = I_1 \omega_1 \wedge \omega_4;\\
&\Omega_6 = A\omega_1 \wedge \omega_4;\\
&\Omega_7 = \omega_1 \wedge (B\omega_3 + C\omega_4);\\
&\Omega_8 = I_2 \omega_2 \wedge \omega_3 \mod\langle\omega_4\rangle.
\end{aligned}$$

The functions I_1, I_2 are fundamental invariants of ω, and with a suitable choice of the section $s\colon \mathcal{E} \to P$, their inverse images have the form

$$\begin{aligned}
s^* I_1 &= -\frac{1}{2} f_0 - \frac{1}{6} f_1 f_2 - \frac{1}{27} f_2^3 + \frac{1}{4} f_{1x} + \frac{1}{6} f_2 f_{2x} - \frac{1}{12} f_{2xx};\\
s^* I_2 &= -\frac{4}{3} f_{2222}.
\end{aligned}$$

Here f_i denote the partial derivatives of f with respect to y_i, $i = 0, 1, 2$, while f_x denotes the total derivative with respect to x:

$$f_x = \frac{df}{dx} = \frac{\partial f}{\partial x} + y_1 \frac{\partial f}{\partial y_0} + y_2 \frac{\partial f}{\partial y_1} + f \frac{\partial f}{\partial y_2}.$$

The third-order ordinary differential equation is contact equivalent to the trivial equation $y''' = 0$ if and only if $I_1 = I_2 = 0$.

The condition $I_1 = 0$ defines C-class of third order ODE's with respect to the contact transformations.

Example 4 (Tanaka (unpublished), Doubrov–Komrakov–Morimoto 1999)

Consider an arbitrary ODE

$$y^{(n+1)} = f(x, y, y', \dots, y^{(n)}), \quad n \geq 3.$$

up to contact transformations.

$$\begin{aligned} E &= \langle \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + \dots + y_n \frac{\partial}{\partial y_{n-1}} + f \frac{\partial}{\partial y_n} \rangle \\ &= \langle dy_0 - y_1\, dx, \dots, dy_{n-1} - y_n\, dx, dy_n - f\, dx \rangle^{\perp} \end{aligned}$$

Model space:

$$\begin{aligned} G &= \operatorname{Sym}(y^{(n+1)} = 0) = GL(2, \mathbb{R}) \leftthreetimes \mathbb{R}^{n+1}; \\ G_0 &= T(2, \mathbb{R}). \end{aligned}$$

The subgroup G_0 is a stabilizer of the action of G on $J^n(\mathbb{R}, \mathbb{R})$ at the point $o = (0, \dots, 0)$.

Theorem. There exists a principal G_0-bundle $\pi \colon P \to \mathcal{E}$ and a Cartan connection ω of type G/G_0 on P, naturally associated with a given ODE of order ≥ 4.

The construction of this connection is algorithmic and can be implemented in any computer algebra system.

Classical Wilczynski invariants

E. Wilczynski, Projective differential geometry of curves and ruled surfaces, Leipzig, Teubner, 1905.

$$y^{(n+1)} + p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0,$$
$$(x, y) \mapsto (\lambda(x), \mu(x)y)$$

Laguerre–Forsyth canonical form:

$$y^{(n+1)} + q_{n-2}(x)y^{(n-2)} + \cdots + q_0(x)y = 0,$$
$$(x, y) \mapsto \left(\frac{ax+b}{cx+d}, \frac{ey}{(cx+d)^n}\right)$$

$$L_k = \sum_{j=1}^{k-2} (-1)^j \frac{(2k-j-1)!(n-k+j)!}{(k-j)!j!} q_{n-k+j}^{(j-1)}$$

$k = 3, \ldots, n+1.$

$$L_3 = p_n'' + \frac{6}{n+1} p_n p_n' + \frac{4}{(n+1)^2} p_n^3 + \frac{6}{n} p_{n-1}' - \frac{12}{n(n+1)} p_n p_{n-1} + \frac{12}{n(n-1)} p_{n-2}.$$

All invariants and C-class equations

Generalized Wilczynski invariants are defined as classical Wilczynski invariants of the linearization:

$$z^{(n+1)} = \frac{\partial f}{\partial y^{(n)}} z^{(n)} + \cdots + \frac{\partial f}{\partial y'} z' + \frac{\partial f}{\partial y} z.$$

All fundamental invariants have the form:

- generalized Wilczynski invariants $L_3, \ldots, L_{n+1}$;
- additional "non-linear invariants":
  - for $n = 3$: $f_{333} = 6f_{233} + f_{33}^2 = 0$;
  - for $n = 4$: $f_{44} = 6f_{234} - 4f_{333} - 3f_{34}^2 = 0$;
  - for $n = 5$: $f_{55} = f_{45} = 0$;
  - for $n \geq 6$: $f_{n,n} = f_{n,n-1} = f_{n-1,n-1} = 0$.

C-class is defined by the condition:

$$L_3 = L_4 = \cdots = L_{n+1} = 0.$$

Summary

(1) ODE's are differential systems of the form $C = V \oplus E$ defined on jet spaces;
(2) to solve the equivalence problem one needs to construct a natural coframe associated with this differential system;
(3) Cartan connections encode the absolute parallelism into a differentail 1-form with values in a Lie algebra; they can be considered as deformations of homogeneous spaces (flat models);
(4) More advanced algebraic techinques are needed for controlling the normalization of the curvalure;
(5) We can construct associated Cartan connections for any ODE of order ≥ 2 and compute the invariants explicitly;
(6) C-classes of equations of order ≥ 3 are defined by vanishing Wilczynski invariants.

REFERENCES

[1] E. Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. France, **52** (1924), pp. 205–241.

[2] S.-S. Chern, *The geometry of the differential equation* $y''' = F(x, y, y', y'')$, Sci. Rep. Nat. Tsing Hua Univ., **4** (1950), pp. 97–111.

[3] B. Doubrov, B. Komrakov, T. Morimoto, *Holonomic systems of differential equations*, Lobachevskij Journal of Mathematics, **3** (1999), pp. 39–71.

[4] B. Doubrov, *Contact trivialization of ordinary differential equations*, Differential geometry and its applications (Opava, 2001), Math. Publ., **3** (2002), pp. 73–84.

[5] M. Fels, *The equivalence problem for systems of second order ordinary differential equations*, Proc. London Math. Soc., **71** (1995), no. 1, pp. 221–240.

[6] A. Gonzàles-Lòpez, *On the linearization of second order ordinary differential equations*, J. Math. Phys., **29** (1989), pp. 341–349.

[7] C. Grissom, G. Thompson, G. Wilkens, *Linearization of second order ordinary differential equations via Cartan's equivalence method*, J. Diff Eq., **77** (1989), pp. 1–15.

[8] L. Hsu, N. Kamran, *Classification of second-order ordinary differentail equations admitting Lie groups of fiber-preserving symmetries*, Proc. London Math. Soc., **58** (1989), pp. 387–416.

[9] S. Kobayashi, *Transformation groups in differential geometry*, Berlin–Heidelberg–New York, Springer–Verlag, 1972.

[10] B. Kostant, *Lie algebra cohomology and the generalized Borel–Weil theorem*, Ann. of Math., **24** (1961), 329–387.

[11] S. Lie, *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, Leipzig, Teubner, 1891.

[12] T. Morimoto, *Un critère pour l'existence d'une connexion de Cartan*, C.R. Acad. Sci. Paris, **308** (1989), pp. 245–248.

[13] P.J. Olver, *Symmetry, invariants, and equivalence*, Neq York, Springer–Verlag, 1995.

[14] H. Sato, A. Y. Yoshikawa, *Third order ordinary differential equations and Legendre connections*, J. Math. Soc. Japan, **50** (1998), pp. 993–1013.

[15] S. Sternberg, *Lectures on differential geometry*, Englewood Cliffs, Printice–Hall, 1964.

[16] N. Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto. Univ., **10** (1970), pp. 1–82.

[17] N. Tanaka, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J., **6** (1979), pp. 23–84.

[18] N. Tanaka, *On geometry and integration of systems of second order ordinary differential equations*, Proc. Symposium on Differential Geometry, 1982, pp. 194–205 (in Japanese).

[19] N. Tanaka, *Geometric theory of ordinary differential equations*, Report of Grant-in-Aid for Scientific Research MESC Japan, 1989.

[20] M.A. Tresse, *Sur les invariants différentiels des groupes continus de transformations*, Acta Math., **18** (1894), pp. 1–88.

[21] M.A. Tresse, *Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre* $y'' = \omega(x, y, y')$, Leipzig, Hirzel, 1896.

[22] K. Yamaguchi, *Geometrization of jet bundles*, Hokkaido Math. J., **12** (1983), pp. 27–40.

[23] K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, Adv. Studies in Pure Math., **22** (1993), pp. 413–494.

[24] E.J. Wilczynski, *Projective differential geometry of curves and ruled surfaces*, Leipzig, Teubner, 1905.