DIFFERENTIAL INVARIANTS OF LIE PSEUDOGROUPS IN MECHANICS OF FLUIDS

Sergey Golovin
Queen's University
Kingston, Ontario K7L 4A5, Canada
e-mail: sergey@mast.queensu.ca; sergey@hydro.nsc.ru

We present bases of differential invariants for Lie pseudogroups admitted by the main models of fluid mechanics. Among them infinite-dimensional parts of symmetry groups of Navier-Stokes equations in general (V.O. Bytev, 1972) and rotationally-symmetric (L.V. Kapitanskij, 1979) cases; stationary gas dynamics equations (M. Munk, R. Prim, 1947); stationary incompressible ideal magnetohydrodynamics (O.I. Bogoyavlenskij, 2000). Applications of the obtained bases to construction of differentially-invariant solutions and group foliations of the differential equations are demonstrated.

Introduction

Let us consider n-dimensional Euclidian space X of independent variables and m-dimensional space U of functions $\mathbf{u}(\mathbf{x})$. Lie group G of point transformations $T : Z \to Z$ acts in the space $Z = X \times U$. The corresponding Lie algebra L is generated by the infinitesimal operators $X = \boldsymbol{\xi} \cdot \partial_{\mathbf{x}} + \boldsymbol{\eta} \cdot \partial_{\mathbf{u}}$. In case of finite-dimensional group G_r the set of L_r generators is finite: $L_r = \{X_1, \ldots, X_r\}$. For the infinite-dimensional group its algebra L contains arbitrary functions in coefficients $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ of infinitesimal operators.

The prolonged space $\underset{k}{Z} = X \times U \times \underset{1}{U} \times \ldots \times \underset{k}{U}$, where $\underset{s}{U}$ is the space of all partial derivatives of the order s: $u^k_{i_1 \ldots i_s} = \frac{\partial^s u^k}{\partial x^{i_1} \ldots \partial x^{i_s}}$, is being taken into consideration. The group action prolongs on space $\underset{k}{Z}$ in the usual way. The corresponding prolonged Lie algebra $\underset{k}{L}$ is generated by the operators

$$\underset{k}{X} = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha} + \zeta^\alpha_i \partial_{u^\alpha_i} + \ldots + \zeta^\alpha_{i_1 \ldots i_k} \partial_{u^\alpha_{i_1 \ldots i_k}}. \tag{1}$$

Here and further (unless specially indicated) on repeating indexes summing is implied; indexes take on the following values: $\alpha = 1, \ldots, m$; $i, i_1, \ldots, i_k = 1, \ldots, n$. The coordinates of prolonged operator (1) are calculated from a known formula

$$\zeta^\alpha_{i_1 \ldots i_s} = D_{i_1} \ldots D_{i_s} (\eta^\alpha - u^\alpha_j \xi^j) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \tag{2}$$

D_i — operator of total differentiation on i-th variable.

The following integer characteristics are important. The first is

$$\nu_k = \dim \underset{k}{Z} = n + m C^n_{n+k}.$$

The second is r_k — general rank of matrix, the rows of which contain the coordinates of all possible operators $\underset{k}{X} \in \underset{k}{L}$:

$$r_k = \max \operatorname{rank} \parallel \boldsymbol{\xi},\ \boldsymbol{\eta},\ \underset{1}{\boldsymbol{\zeta}},\ \ldots,\ \underset{k}{\boldsymbol{\zeta}} \parallel .$$

The sequence of numbers r_k is nondecreasing: $r_0 \leq r_1 \leq \ldots \leq r_k \leq \ldots$. For the finite-dimensional Lie groups the sequence r_k on the finite prolongation always reaches the maximal value r, equal to group dimension. For the infinite-dimensional groups the sequence r_k, generally speaking, in not limited.

Definition 1. A differential invariant of k-th order of group G is such a function $\underset{k}{J}(\mathbf{x},\, \mathbf{u},\, \underset{1}{\mathbf{u}}, \dots, \underset{k}{\mathbf{u}})$, that for any transformation $T \in G$ the following holds

$$\underset{k}{J}(T\mathbf{x},\, T\mathbf{u},\, \underset{1}{T}\underset{1}{\mathbf{u}}, \dots, \underset{k}{T}\underset{k}{\mathbf{u}}) = \underset{k}{J}(\mathbf{x},\, \mathbf{u},\, \underset{1}{\mathbf{u}}, \dots, \underset{k}{\mathbf{u}})$$

The differential invariants of the zeroth order are finite invariants of group G. For the differential invariants the usual invariance criterion is valid:

Lemma 1. *Function* $\underset{k}{J}(\mathbf{x},\, \mathbf{u},\, \underset{1}{\mathbf{u}}, \dots, \underset{k}{\mathbf{u}})$ *is the differential invariant of group* G *iff for any* $\underset{k}{X} \in \underset{k}{L}$ *is fulfilled*

$$\underset{k}{X}\,\underset{k}{J} = 0.$$

There are exactly $\nu_k - r_k$ functionally independent differential invariants of the order not higher than k. In case $\lim\limits_{k\to\infty}(\nu_k - r_k) = \infty$, group G has infinitely many functionally independent differential invariants.

Definition 2. Operator $\delta = \lambda^i D_i$ is an operator of invariant differentiation iff for any differential invariant $\underset{k}{J}$ function $\delta\,\underset{k}{J} = \lambda^i\,(D_i\,\underset{k}{J})$ is also differential invariant.

The following lemma gives the criteria for the operators of invariant differentiation.

Lemma 2. *The operators of invariant differentiation generate Lie algebra over the field of differential invariants of group* G*. Any operator, commutating with all the operators* $\underset{\infty}{X} \in \underset{\infty}{L}$ *is the operator of invariant differentiation.*

The consequence of lemma 2 are the equations for coefficients λ^i of the operator of invariant differentiation:

$$\forall X \in L : \quad (\lambda^i D_i)\, \xi^j = \underset{\infty}{X}\, \lambda^j, \quad j = 1, \ldots, n. \tag{3}$$

Here ξ^j is a coefficient at ∂_{x^j} in the operator X.

It is known (A.P. Chupakhin, 2004), that the algebra of invariant differentiation operators can always be transformed to commutative one. The central fact of differential invariants theory is a theorem on the existence of finite differential invariants basis.

Theorem 1. *For any group G there is a finite differential invariants basis, i.e. such finite set of scalar differential invariants through which the arbitrary invariant of group G is obtained by the use of finite number of functional operations and invariant differentiations.*

The following statement underlines the importance of differential invariants basis.

Lemma 3. *The basis of differential invariants together with invariant differentiation operators of group G uniquely determine group G itself, i.e. allows one to construct the determining equations of G.*

In the next sections the calculations of differential invariants bases for infinite-dimensional groups admitted by different models of continuous media are presented.

1 Rotationally-symmetrical motions of a viscous fluid

The Navier — Stokes equations are observed in cylindrical coordinate system (r, θ, z). It is supposed that the components of velocity

(v_r, v_θ, v_z) and pressure p do not depend on polar angle θ. The continuity equation allows to introduce the stream function ψ by the relations

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{4}$$

For more convenience function $w = r v_\theta$ is used instead of component of velocity v_θ. Excluding pressure from momentum equations by cross differentiation, we obtain a definite system for ψ and w

$$\begin{gathered} D\,\psi_t - \nu D^2\psi - r\frac{\partial(\psi, r^{-2}D\,\psi)}{\partial(r,z)} - \frac{1}{r^2}\frac{\partial}{\partial z}(w^2) = 0, \\ w_t - \nu D\, w - \frac{1}{r}\frac{\partial(\psi, w)}{\partial(r,z)} = 0. \end{gathered} \tag{5}$$

Here $D = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$; $\frac{\partial(\psi,w)}{\partial(r,z)} = \psi_r w_z - \psi_z w_r$. The group of point transformations admitted by the system (5) is generated by the following operators (L.V. Kapitanskij, 1979)

$$\begin{gathered} X_1 = \partial_t, \quad X_2 = 2t\partial_t + r\partial_r + z\partial_z + \psi\partial_\psi, \\ X_3(\tau) = \tau(t)\partial_z - \frac{1}{2}r^2\,\dot\tau(t)\partial_\psi, \quad X_4(\sigma) = \sigma(t)\partial_\psi. \end{gathered} \tag{6}$$

Here $\tau(t)$ and $\sigma(t)$ are arbitrary smooth functions. Below, the bases of differential invariants for the infinite-dimensional group $\{X_3(\tau),\ X_4(\sigma)\}$, and also for the complete group (6) are calculated.

It is obvious that w is an invariant of group (6). All differential invariants, depending on derivatives of w, are obtained by the application of invariant differentiations to w necessary number of times. Therefore, the group action is considered below only in space $Z = X \times Y = \mathbb{R}^3(t,r,z) \times \mathbb{R}(\psi)$. Here $n = 3$, $m = 1$.

Theorem 2. *Operators of invariant differentiation for the Lie pseudogroup group, generated by operators* $\{X_3(\tau),\ X_4(\sigma)\}$ *are*

$$\delta_1 = rD_t - \psi_r D_z, \quad \delta_2 = D_r, \quad \delta_3 = D_z. \tag{7}$$

Its basis of differential invariants is

$$t, \quad r, \quad w, \quad \psi_z, \quad \psi_{rr} - \frac{1}{r}\psi_r. \tag{8}$$

The differential invariants for complete group (6) are produced from those obtained in Theorem 2.

Theorem 3. *The invariant differentiation operators for complete group* (6) *are the following:*

$$\delta_1 = r^2\, D_t - r\, \psi_r\, D_z, \quad \delta_2 = r\, D_r, \quad \delta_3 = r\, D_z.$$

The basis of differential invariants consists of three invariants

$$j_1 = w, \quad j_2 = \psi_z, \quad j_3 = r\, \psi_{rr} - \psi_r.$$

2 The space motions of a viscous liquid

The Navier — Stokes equations for space motions of a viscous liquid are recorded by

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\triangle\mathbf{u}, \quad \operatorname{div}\mathbf{u} = 0. \tag{9}$$

It is known (V.O. Bytev, 1972), that equations (9) admit infinite-dimensional transformations group. Its finite-dimensional part is generated by time transition, three rotations and dilatation. Lie algebra of operators

$$Y_i(\varphi_i(t)) = \varphi_i\partial_{x^i} + \dot\varphi_i\partial_{u^i} - x^i\ddot\varphi_i\partial_p, \ (i = 1, 2, 3); \quad Z = \sigma(t)\partial_p \tag{10}$$

(there is no summing on index i) corresponds to infinite-dimensional part of the admitted group. Functions $\varphi_i(t)$ and $\sigma(t)$ are arbitrary.

Theorem 4. *The invariant differentiation operators for group generated by operators* (10) *are as follows*

$$\delta_0 = D_t + uD_x + vD_y + wD_z,$$

$$\delta_1 = D_x, \quad \delta_2 = D_y, \quad \delta_3 = D_z.$$

The basis of differential invariants consists of 13 invariants:

$$t, \ \ \nabla u, \ \ \nabla v, \ \ \nabla w, \ \ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p.$$

This very result is valid also for Euler equations, describing the motion of an ideal liquid ($\nu = 0$).

3 The stationary gas dynamics.

Equations, describing three-dimensional stationary motions of nonviscous nonheat-conducting gas are considered.

$$(\mathbf{u}\cdot\nabla)\mathbf{u}+\rho^{-1}\nabla p = 0, \ \ (\mathbf{u}\cdot\nabla)\rho+\rho\,\mathrm{div}\mathbf{u} = 0, \ \ (\mathbf{u}\cdot\nabla)S = 0. \quad (11)$$

Gas state equation has a separated form: $p = f(S)g(\rho)$. Equations (11) apart of finite group admit infinite-dimensional Munk&Prim transformation (M. Munk, R. Prim, 1947).

$$X = -m(\mathbf{x})\,(\mathbf{u}\,\partial_{\mathbf{u}}(\mathbf{x}) + 2\,\rho\,\partial_\rho + 2\,S\,\partial_S), \quad \mathbf{u}\cdot\nabla m(\mathbf{x}) = 0. \quad (12)$$

Below, the differential invariants basis and invariant differentiation operators for the transformation (12) are formed.

Theorem 5. *The operators of invariant differentiation for transformation* (12) *coincide with total differentiation with respect to independent variables*

$$\delta_1 = D_x, \quad \delta_2 = D_y, \quad \delta_3 = D_z.$$

The differential invariants basis of transformation (12) *may be chosen in the form:*

$$x, \quad y, \quad z, \quad \mathbf{u}\sqrt{\rho}, \quad \frac{S}{\rho}, \quad \frac{(\mathbf{u}\cdot\nabla)\rho}{\sqrt{\rho}}. \tag{13}$$

4 The transonic gas motions

One of spread models, used in description of transonic gas flows, is the Karman-Guderley equation:

$$-\varphi_x\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0. \tag{14}$$

Function $\varphi(x, y, z)$ gives the potential of small perturbations of the gas flow, moving uniformly with critical velocity along the Ox axis. Direct calculation shows that equation (14) admits infinite-dimensional algebra of transformations $L_6 \oplus L_\infty$. Its finite part L_6 is generated by the following operators

$$\begin{gathered} Y_1 = \partial_x, \quad Y_2 = \partial_y, \quad Y_3 = \partial_z, \quad Y_4 = z\partial_y - y\partial_z, \\ Y_5 = y\partial_y + z\partial_z - 2\varphi\partial_\varphi, \quad Y_6 = x\partial_x + 3\varphi\partial_\varphi. \end{gathered} \tag{15}$$

The operator

$$X_\infty = f(y, z)\partial_\varphi, \quad \triangle f(y, z) = 0 \tag{16}$$

corresponds to the infinite-dimensional part L_∞ of admitted algebra. There are no any nontrivial contact transformations, admitted by equation (14).

Theorem 6. *The differential invariants basis for transformation, generated by operator X_∞ may be selected in the following way*

$$x,\ \ y,\ \ z,\ \ \varphi_x,\ \ \varphi_{yy}+\varphi_{zz}. \tag{17}$$

The operators of invariant differentiation are operators of total differentiation on independent variables x, y, z.

5 Magnetohydrodynamics equilibria

The system of ideal magnetohydrodynamics equilibria equations has the following form

$$\begin{aligned} &\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\mu^{-1}\,\mathbf{B}\times\operatorname{rot}\mathbf{B}+\nabla p=0,\\ &\operatorname{div}(\rho\,\mathbf{u})=0,\quad \operatorname{div}\mathbf{B}=0,\quad \operatorname{rot}(\mathbf{u}\times\mathbf{B})=0. \end{aligned} \tag{18}$$

Here $\mathbf{u}$ is the velocity of plasma particles, $\mathbf{B}$ is the magnetic vector field, μ is a constant magnetic permeability, ρ is density and p is the pressure. We assume that fluid is incompressible, which implies $\operatorname{div}\mathbf{u}=0$. From (18) it follows that $(\mathbf{u}\cdot\nabla)\rho=0$, i.e. density conserves along streamlines. It is known (O.I. Bogoyavlenskij, 2000) that system (18) under assumption of incompressibility admits an infinite-dimensional group of transformations

$$\begin{aligned} &\mathbf{B}_1=b(\mathbf{x})\,\mathbf{B}+c(\mathbf{x})\sqrt{\mu\rho(\mathbf{x})}\,\mathbf{u},\quad \mathbf{u}_1=\frac{c(\mathbf{x})}{a(\mathbf{x})\sqrt{\mu\rho(\mathbf{x})}}\mathbf{B}+\frac{b(\mathbf{x})}{c(\mathbf{x})}\mathbf{u},\\ &\rho_1(\mathbf{x})=a^2(\mathbf{x})\rho(\mathbf{x}),\quad p_1=Cp+(C\mathbf{B}^2-\mathbf{B}_1^2)/(2\mu). \end{aligned} \tag{19}$$

Here $C=\text{const}$, $a(\mathbf{x})$, $b(\mathbf{x})$ and $c(\mathbf{x})$ are arbitrary functions, constant along streamlines and magnetic field lines and such that $b^2(\mathbf{x})-c^2(\mathbf{x})=C$.

Let us introduce the following variables

$$\mathbf{v} = \mathbf{u}\sqrt{\mu\rho(\mathbf{x})}, \quad P = \mu p + \frac{1}{2}\mathbf{B}^2 \tag{20}$$

After some straightforward transformations system (18) can be written in the equivalent form

$$\begin{aligned} &(\mathbf{v}\cdot\nabla)\,\mathbf{v} - (\mathbf{B}\cdot\nabla)\,\mathbf{B} + \nabla P = 0, \\ &(\mathbf{B}\cdot\nabla)\mathbf{v} = (\mathbf{v}\cdot\nabla)\mathbf{B}, \\ &\operatorname{div}\mathbf{v} = 0, \quad \operatorname{div}\mathbf{B} = 0. \end{aligned} \tag{21}$$

The equation $(\mathbf{u}\cdot\nabla)\rho = 0$ for density ρ splits from the system (21), therefore it can be observed separately.

Theorem 7. *The admissible group of the system* (21) *is a semidirect sum of the 7-dimensional group G_7 and infinite-dimensional normal subgroup G_∞: $G = G_7 \oplus G_\infty$. The corresponding Lie algebras are generated by the following infinitesimal operators*

$$\begin{array}{ll} & X_i = \partial_{x^i}, \quad i = 1,2,3 \\ & R_i = \varepsilon_{ijk}(x^j\partial_{x^k} + u^j\partial_{u^k} + B^j\partial_{B^k}), \quad i = 1,2,3 \\ G_9: & S_1 = x^i\partial_{x^i}, \\ & S_2 = v^i\partial_{v^i} + B^i\partial_{B^i} + 2P\partial_P \\ & P = \partial_P \\ G_\infty: & <\varphi>= \varphi(\mathbf{x})(B^i\partial_{v^i} + v^i\partial_{B^i}) \end{array}$$

Here summation over repeated indexes is performed, ε_{ijk} is the permutation tensor. The function $\varphi(\mathbf{x})$ is an arbitrary function satisfying equations

$$(\mathbf{v}\cdot\nabla)\varphi = 0, \quad (\mathbf{B}\cdot\nabla)\varphi = 0. \tag{22}$$

i.e. constant along streamlines and magnetic field lines. The system (21) *also admits the following discrete transformations*

$$\begin{aligned} \varepsilon_1 : &\quad \mathbf{x}_1 = -\mathbf{x} \\ \varepsilon_2 : &\quad \mathbf{v}_1 = -\mathbf{v} \\ \varepsilon_3 : &\quad \mathbf{B}_1 = -\mathbf{B} \\ \varepsilon_4 : &\quad \mathbf{v}_1 = \mathbf{B}, \quad \mathbf{B}_1 = \mathbf{x}, \quad P_1 = -P. \end{aligned}$$

Our goal is to construct basis of differential invariants of group G_∞. The finite invariants of G_∞ are

$$\mathbf{x}, \quad a^{ij} = v^i v^j - B^i B^j, \quad b^{ij} = v^i B^j - v^j B^i, \quad P. \tag{23}$$

There are only 9 functionally independent invariants among them. The relations between the invariants are

$$\begin{aligned} &a^{ij} = a^{ji}, \quad b^{ij} = -b^{ji}, \quad (b^{ij})^2 = (a^{ij})^2 - a^{ii}a^{jj}, \\ &(a^{12} - b^{12})(a^{23} - b^{23})(a^{31} - b^{31}) = a^{11}a^{22}a^{33}. \end{aligned} \tag{24}$$

Part of the first-order differential invariants is given by

$$c^i = v^j v^i_j - B^j B^i_j, \quad d^i = v^j B^i_j - B^j v^i_j, \quad i = 1, 2, 3. \tag{25}$$

Theorem 8. *Basis of differential invariants of the group* G_∞ *consists of the following invariants*

$$\mathbf{x}, \quad a^{11}, \quad a^{12}, \quad a^{13}, \quad a^{22}, \quad a^{23}, \quad P, \quad c^1, \quad d^1. \tag{26}$$

Operators of invariant differentiation are complete derivativaes with respect to independent variables $\mathbf{x}$*.*

6 Differentially invariant solutions

There are a number of known examples of comparatively simple solutions of the Navier — Stokes equations that do not have explicit group interpretation. One of them is Karman's solution

(Th. Karman, 1921) describing stationary liquid flow between two rotating disks. Another similar solution, is presented in Aristov's work (S.N. Aristov, 2001). It qualitatively describes the flow, observed when stirring tea in a glass. It turns out that both these solutions can be described in terms of differential invariants, and can be interpreted as differentially invariant ones.

Indeed, Karman's solution in a cylindrical coordinate system has the following form

$$v_r = rU(z), \quad v_\varphi = rV(z), \quad v_z = W(z)$$

or with the use of functions ψ and w (system (5)),

$$\psi = r^2\psi_0(z), \quad w = r^2 w_0(z). \tag{27}$$

Here, U, V, W, ψ_0, w_0 are functions, subject to the definition from the system of ordinary differential equations.

Aristov's solution has a similar representation

$$v_r = U(r), \quad v_\varphi = zV(r), \quad v_z = zW(r)$$

In terms of functions ψ and w we have

$$\psi = z\psi_0(r), \quad w = zw_0(r). \tag{28}$$

Both of solutions are usually interpreted in literature as self-similar ones. However, the groups, admitted by the Navier-Stokes equations (5) or (9), do not contain the required dilatation transformation. In general, representations (27), (28) cannot be written in terms of finite invariants of the group, admitted by equations (5) or (9).

The algorithm of obtaining both of classical invariant and partially invariant solutions consists in overdetermination of original system of equations by additional relations between finite invariants. The same principle is put into the base of algorithm of differentially invariant solutions construction. Namely, in order to obtain differentially invariant solutions it is necessary to overdetermine the original system by the relations between differential invariants. Then, it is required to complite an obtained system to involution. Making this system involute is considerably simplified in case when an additional relation can be integrated.

The solution (28) is obtained by the giving of dependence of the most general type between differential invariants (8) both of first and zero orders:

$$\psi_z = \Phi(r,t)$$

Hence, the linear dependence ψ from z follows. A necessary representation for w is obtained automatically from the analysis of equations (5).

In order to obtain Karman's solution (27) it is necessary to consider differential invariants of both second and zero orders to be functionally dependent.

$$\psi_{rr} - r^{-1}\psi_r = \Phi(r,t)$$

integrating of this relation we obtain the representation for the stream function ψ:

$$\psi = r^2 f(z,t) + g(z,t) + h(r,t) \tag{29}$$

which in particular case $g = h = 0$ gives the representation (27).

7 Group foliation

The problem of group foliation (or group stratification) of a given system of equations was first set by Sophus Lie. It naturally arises from the following observations. Suppose we have a system of differential equations E and a group G of transformations, preserving the system E. According to definition, the group G transforms any solution U: $\mathbf{u} = \mathbf{u}(\mathbf{x})$ of system E into some another solution of E. Thus, in the set of all solutions of system E there is the equivalence relation: Two solutions U and U' are equivalent if they are connected by group G transformation, i.e. $U' = T\,U$ with the suitable transformation $T \in G$. Each equivalence class of solutions is formed by the orbit of some solution under all possible transformations of group G. The problem of group foliation is formulated as follows: For a given system of equations E and its admissible group G it is required to split E into the equivalent union of the following two subsystems. The first subsystem denoted as AG is automorphic, i.e. any two solutions of this subsystem are equivalent. In other words, any solution of AG could be obtained from any other solution of AG by the action of admissible group G. The second subsystem denoted as RS is the resolving one. Group G acts trivially on the solutions of RG, which means that solutions of RG distinguish orbits of non-equivalent solutions of system E. Union of the automorphic AG and resolving RG subsystems is equivalent to the initial system E in sense that locally these two systems have the same set of solutions.

In some cases the group foliation gives simple resolving system, which inherits the finite part of the original admissible group. It allows one to give a constructive description of its invariant solutions. Any solution of the resolving system chooses some particular orbit described by the automorphic subsystem. It is enough to choose any solution of the automorphic system obtain the solution of the initial differential equations. However, in many cases the group foliation leads to complication of the original system of equations and therefore is being mostly of academic interest. Below we give examples of group foliations for some mathematical models of fluid mechanics with infinite-dimensional symmetry group.

The main observation in construction of the group foliation for given system of differential equations is that both automorphic and resolving system should relate differential invariants of the group only. We will use “direct” approach by L.V. Ovsiannikov. Let us start from the stationary gas dynamics equations given in section 3. The result is formulated in the following statement.

Theorem 9. *Group stratification of stationary gas dynamics equations* (11) *with respect to Munk&Prim transformation* (12) *consists of resolving system*

$$(\mathbf{w}\cdot\nabla)\,\mathbf{w}+\nabla F(\sigma)=0,\quad \operatorname{div}(\sigma^{-2}\mathbf{w})=0. \tag{30}$$

and automorphic system

$$\mathbf{u}=S^{-1/2}\mathbf{w},\quad S=\rho\sigma^2,\quad R=-2\frac{(\mathbf{w}\cdot\nabla)\sigma}{\sigma^2},\quad D\rho=\rho^{1/2}\,R. \tag{31}$$

For any solution of resolving system (30) *the original unknown functions are restored according to formulae* (31)*.*

We would like to point out that the resolving system is simpler then original equations (11) because it contains less unknown functions. Another example related to the Karman-Guderley equation (14).

Theorem 10. *Group stratification of equation* (14) *with respect to infinite-dimensional group* (16) *has the following form.*

An automorphic system consists of two equations

$$\begin{aligned} \varphi_x &= a, \\ \varphi_{yy} + \varphi_{zz} &= a\, a_x. \end{aligned} \tag{32}$$

Resolving equation determines function $a(x, y, z)$:

$$-a a_{xx} - a_x^2 + a_{yy} + a_{zz} = 0. \tag{33}$$

Another example of relatively simple group foliation related to the magnetohydrodynamics equilibria (21). One can construct group foliation of equations (21) with respect to the admissible pseudogroup G_∞.