Ambient connections realising conformal Tractor holonomy

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Conformal Structures and Tractor bundles

A conformal structure $[g]$ on a manifold M can be described as the ray sub-bundle in the bundle of metrics,

$$\mathcal{Q} = \cup_{p\in M}\mathcal{Q}_p, \text{ with } \quad \mathcal{Q}_p = \mathbb{R}^+ \cdot g_p,$$

with a natural $\mathbb{R}^+$-action. Associated to this there are *density bundles*,

$$\mathcal{E}[w] \; := \; \mathcal{Q} \times_{\rho^w} \mathbb{R}, \qquad TM[w] := \mathcal{E}[w] \otimes TM, \quad \text{etc.},$$

where $\rho^w(t) = t^w$. Then a conformal structure $[g]$ can be considered as a metric $\mathbf{g}$ on $TM[-1]$, i.e. a section $\mathbf{g} \in \Gamma(\odot^2 T^*M[2])$.

The geometric vector bundle in conformal geometry is the *normal conformal Tractor bundle*. It is a vector bundle which is associated to the principle fibre bundle on which the normal conformal Cartan connection lives. Given a metric $g \in [g]$ it is given as

$$\mathcal{T} \;\; \simeq \;\; \mathcal{E}[1] \oplus TM[-1] \oplus \mathcal{E}[-1],$$

$$\nabla^{\mathcal{T}}_X \begin{pmatrix} \tau \\ Y \\ \sigma \end{pmatrix} \;=\; \begin{pmatrix} \nabla^g - \mathbf{g}(X,Y) \\ \nabla^g_X Y + \sigma \otimes X + \tau \otimes \mathsf{P}^g(X) \\ \nabla^g_X \sigma - \mathsf{P}^g(X,Y) \end{pmatrix}$$

P^g Schouten tensor of g, resp. $P^g : TM \to TM[-2]$ dualised with $\mathbf{g}$. Parallel sections correspond to Einstein scales. [Thomas '26+32, Eastwood/Bailey/Gover '94]

Theorem. [Čap/Gover '02, '03]
Let $\mathcal{T}$ be a vector bundle over M^{p+q} of rank $p+q+2$.

1. If $\mathcal{T}$ admits a metric h of signature $(p+1, q+1)$, a vector bundle injection $\mathcal{E}[-1] \hookrightarrow \mathcal{T}$ with image $\mathcal{T}^1$, and a covariant derivative $\overrightarrow{\nabla}$ such that
   (a) $\mathcal{T}^1$ is light-like,
   (b) $\overrightarrow{\nabla} h = 0$, and
   (c) $\forall p \in M$, $\forall \sigma \in \Gamma(\mathcal{T}^1)$ with $\sigma(p) \neq 0$, the map $\overrightarrow{\nabla}\sigma : T_xM \to \mathcal{T}_x$ is non-degenerate.

   Then $(\mathcal{T}^1)^\perp/\mathcal{T}^1 \simeq TM[-1]$ and $(\mathcal{T}, h, \overrightarrow{\nabla})$ is a standard tractor bundle for the conformal structure defined by the restriction of h onto $\left(\mathcal{T}^1\right)^\perp/\mathcal{T}^1$.

2. $(\mathcal{T}, \mathcal{T}^1, \overrightarrow{\nabla}, h)$ is *normal* $\iff$ its curvature $\mathcal{R}$ maps $\mathcal{T}^1$ onto $\mathcal{T}^1$ and the Ricci contraction of $W \in \Lambda^2 T^*M \otimes End(TM[-1])$, defined by $W(X,Y)[Z] = [\mathcal{R}(X,Y)Z]$ vanishes. With these normality conditions $(\mathcal{T}, \mathcal{T}^1, h, \overrightarrow{\nabla})$ is uniquely determined by the conformal structure up to isomorphism.

We are interested in the holonomy of the normal conformal Tractor connection,

$$Hol_p(\mathcal{T}, \nabla^{\mathcal{T}}) := \{\mathcal{P}_\gamma^{\nabla^{\mathcal{T}}} \mid \gamma \text{ loop around } p\}.$$

Ambient Metrics

Let $\pi : \mathcal{Q} \to M^n$ be a conformal structure.
$\widetilde{M}^{n+2}$ is an **ambient manifold** if $\exists$ free $\mathbb{R}^+$-action $\widetilde{\varphi}$ on M, and an embedding $\iota : \mathcal{Q} \to \widetilde{M}$:

$$\begin{array}{ccc} \mathbb{R}^+ \times \mathcal{Q} & \stackrel{\varphi}{\longrightarrow} & \mathcal{Q} \\ {\scriptstyle id\times\iota}\downarrow & \circlearrowright & \downarrow{\scriptstyle \iota} \\ \mathbb{R}^+ \times \widetilde{M} & \stackrel{\widetilde{\varphi}}{\longrightarrow} & \widetilde{M}. \end{array}$$

Let $F(x) = \frac{d}{dt}\left(\widetilde{\varphi}(\mathrm{e}^t, x)\right)|_{t=0}$ be the fundamental vector field of the $\mathbb{R}^+$-action, $\phi := h(F, .)$.
A metric h on $\widetilde{M}$ is an **ambient metric** if

1. $\mathcal{L}_F h = 2h$,
2. $(\iota^* h)_{g_x} = g_x\left(d\pi(.), d\pi(.)\right)$, $\forall\ g_x \in \mathcal{Q}$.
3. $Ric^h = 0$.

Existence and Uniqueness (up to $\mathbb{R}^+$-equivariant diffemorphism) of a metric with 1,2 and 3 if $[g]$ analytic: always for n odd, for n even replace 3 by 3': along $\mathcal{Q}$, Ric vanishes to order $(n-4)/2$ and $Ric|_{T\mathcal{Q}}$ vanishes to order $(n-2)/2$. [Fefferman/Graham '85]. [Čap/Gover '03] considered ambient metrics without 3 or 3'.

Ricci-flat ambient metrics if the conformal class contains an Einstein metric:

Let (M,c) be a conformal manifold, $g \in c$ Einstein, i.e. $Ric^g = \frac{scal}{n} g$. Two cases:

<u>(1) $scal = 0$:</u> "degenerate cone"

$$\begin{aligned} \widetilde{M} &:= \mathbb{R} \times M \times \mathbb{R}^+ \{x,p,z\} \\ \mathcal{Q} &:= \{x = 0\} \\ h &:= 2dxdz + z^2 g. \end{aligned}$$

<u>(2) $scal \neq 0$:</u> "metric cone"

$$\begin{aligned} \widetilde{M} &:= \mathbb{R} \times M \times \mathbb{R}^+ = \{(x,p,t)\} \\ \mathcal{Q} &:= \{x = -t\} \\ h &:= \frac{(1-n)n}{scal} dx^2 + \underbrace{\frac{n(n-1)}{scal} dt^2 + t^2 g.}_{\text{cone metric}} \end{aligned}$$

<u>In both cases:</u> [L '05], [Leitner '04, Armstrong '05]

- $\nabla^h \frac{\partial}{\partial x} = 0$,
- $Ric^h = 0$,
- $Hol(\widetilde{M}, \nabla^h) = Hol(\mathcal{T}, \nabla^{\mathcal{T}})$.

For a general ambient metric: $Hol(M, \nabla^h) \neq Hol(\mathcal{T}, \nabla^{\mathcal{T}})$!

Aim: For $(\widetilde{M}, h)$ an ambient metric for (M, c), find a connection $\widetilde{\nabla}$ such that:

- $\widetilde{\nabla} = \nabla^h$ in the Einstein case,
- always $Hol(M, \nabla^h) = Hol(\mathcal{T}, \nabla^{\mathcal{T}})$,
- compatibility with the ambient metric.

$\rightsquigarrow$ **ambient connection:** [Armstrong/L '06]

$\widetilde{\nabla}$ is a connection on $(\widetilde{M}, h)$ with torsion T such that

(1) $\widetilde{\nabla} h = 0$,

(2) $\mathcal{L}_F T^* \ = \ 2T^*$, i.e. $\mathcal{L}_F T = 0$.

(The second condition ensures: if X and Y vector fields of homogeneity v and w, then $\widetilde{\nabla}_X Y$ is of homogeneity $v + w$)

Normal tractor bundle defined by ambient objects:

1. $\mathcal{Q} \subset \widetilde{M}$ defines a bundle $\mathcal{T}$: $\mathbb{R}^+$ acts on $T\widetilde{M}|_{\mathcal{Q}}$ by
$$\varphi^*(t, X_q) \;:=\; t^{-1}\left(d\varphi(t,.)\right)_q(X_q).$$
$$\Rightarrow \;\mathcal{T} := \left(T\widetilde{M}|_{\mathcal{Q}}\right)/\mathbb{R}^+ \;\leftarrow (n+2)\text{-dim. fibres}$$
$$\downarrow$$
$$M \simeq \mathcal{Q}/\mathbb{R}^+$$
Note that $\Gamma(\mathcal{T}) \simeq \Gamma^{-1}(T\widetilde{M}|_{\mathcal{Q}})$. Hence, for $X \in \Gamma(\mathcal{T})$, denote by $\widetilde{X}$ the vector field along $\mathcal{Q}$ of homogeneity -1.

2. $\mathcal{T}^1$ is given by the fundamental vector field F.

3. h defines a metric $h^{\mathcal{T}}$ on $\mathcal{T}$: for $\widetilde{X}$ and $\tilde{Y}$ homogeneous of degree -1, $h(X,Y)$ is constant along the orbits.

4. If $d\phi|_{\mathcal{Q}} = 0$, then an ambient connection $\widetilde{\nabla}$ with $T|_{\mathcal{Q}} = 0$ defines a connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$ via $\widetilde{\nabla^{\mathcal{T}}_X Y} := \widetilde{\nabla}_X \tilde{Y}$ such that $(\mathcal{T}, h^{\mathcal{T}}, \nabla^{\mathcal{T}})$ is a **conformal standard tractor bundle**.

5. If in addition $T(F,.) = T^*(.,.,F) = 0$, then $(\mathcal{T}, h^{\mathcal{T}}, \nabla^{\mathcal{T}})$ is **normal** $\Leftrightarrow \iota^*\widetilde{Ric} = 0$.

Example of an ambient connection with Cotton–York tensor as torsion:

Set $\widetilde{M} := (-\varepsilon, \varepsilon) \times M \times \mathbb{R}^+ \ni (s, x, q)$,

$$\begin{aligned} \mathcal{Q} \ \ni \ t^2 g_x \ &\overset{\iota}{\longmapsto} \ (0, x, t) \ \in \widetilde{M}, \\ \varphi_t(s, x, q) \ &= \ (ts, x, tq), \\ F \ &= \ sS + qQ \end{aligned}$$

Now, fix a metric $g \in c$ with Schouten tensor P, define:

$$\Gamma(TM) \ \ni \ X \ \longmapsto \ \widetilde{X} = (s\mathsf{P} + q\mathsf{Id})^{-1}(X) \ \in \ \Gamma(T\widetilde{M})$$

$\uparrow$

homogeneous of degree -1,

not defined for $-\frac{s}{q} =$ (eigen value of P)$^{-1}$,

but defined on $\mathcal{Q}$ and for ε small

Define an ambient metric,

$$\begin{aligned} h(\widetilde{X}, \widetilde{Y}) \ &= \ g(X, Y) \\ h(S, Q) \ &= \ 1, \text{ for } X, Y \in TM, \quad \text{and 0 otherwise.}, \end{aligned}$$

and an ambient connection,

$$\widetilde{\nabla}_X \widetilde{Y} \ = \ \widetilde{\nabla_X Y} - g(X, Y)S - \mathsf{P}(X, Y)Q,$$

$$\begin{aligned} \text{and} \quad \widetilde{\nabla}_X Q \ &= \ \widetilde{X} \ = \ \widetilde{\nabla}_Q X, \\ \widetilde{\nabla}_X S \ &= \ \widetilde{\mathsf{P}(X)} \ = \ \widetilde{\nabla}_S X \end{aligned}$$

and all the other Q and S terms being zero.

Then $d\phi = 0$, $\widetilde{\nabla} h = 0$ and

$$\boxed{T(X,Y) = s\ \widetilde{CY(X,Y)},}$$

for $X, Y, Z \in TM$, and zero otherwise, where

$$CY(X,Y) = (\nabla_X \mathsf{P})(Y) - (\nabla_Y \mathsf{P})(X)$$

is the Cotton–York tensor. In particular, $(\mathcal{L}_F T^*) = 2T^*$, $T|_{\mathcal{Q}} = 0$, $T(F,.,.) = T^*(.,.,F) = 0$.

<u>Curvature:</u>

$\widetilde{\mathcal{R}}(X,Y)\tilde{Z} \;=\; W\widetilde{(X,Y)}Z - CY(X,Y,Z) \cdot Q$, and thus

$$\left.\begin{array}{rcl} \widetilde{Ric}(X,S) & = & \sum_{i=1}^n CY(\tilde{E}_i, X, E_i) \\ \widetilde{Ric}(X,\tilde{Y}) & = & \sum_{i=1}^n g(W(\tilde{E}_i, X)Y, E_i) \end{array}\right\} \quad i.e. \quad \widetilde{Ric}|_{\mathcal{Q}} = 0.$$

$\Rightarrow$ $(\widetilde{M}, h, \widetilde{\nabla})$ defines the normal conf. tractor bundle.

Furthermore, for $X \in TM$

$$\widetilde{\nabla}_S \widetilde{X} = 0 \quad \text{and} \quad \widetilde{\nabla}_Q \widetilde{X} = 0$$

$\Rightarrow$

Theorem. [Armstong/L '06] The holonomy of $\widetilde{\nabla}$ is generated by paths in M, and hence

$$Hol_{(0,x,1)}(T\widetilde{M}, \widetilde{\nabla}) = Hol_{(0,x,1)}(T\widetilde{M}|_M, \widetilde{\nabla}) = Hol_x(\mathcal{T}, \nabla^{\mathcal{T}})$$

Special cases:

1. $[g]$ contains a C-space, i.e. $CY = 0$. Then $\widetilde{\nabla} = \nabla^h$ is the Levi-Civita connection of h, but $\widetilde{Ric}$ not necessarily zero.

   Corollary: If $CY = 0$, i.e. $\widetilde{\nabla}$ torsion-free $\Rightarrow$ Tractor holonomy algebra is a **Berger algebra**.

   This is a geometric proof of our result in [L '05].

2. $g \in [g]$ with $Ric^g = 0$, then $\widetilde{X} = \frac{1}{q} X$, $\widetilde{\nabla} = \nabla^h$, $Ric^h = 0$ and

$$h \; = \; 2dsdq + q^2 g.$$

3. $[g]$ contains an Einstein space, then $\widetilde{X} = \left(s\frac{scal}{2n(n-1)} + q\right)^{-1} X$, $\widetilde{\nabla} = \nabla^h$, $Ric^h = 0$ and

$$h = 2dsdq + \left(s\frac{scal}{2n(n-1} + q\right)^2 g.$$

   Changing coordinates gives the familiar form

$$h = \frac{(1-n)n}{scal} dx^2 + \frac{n(n-1)}{scal} dt^2 + t^2 g.$$

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