Separation of variable theory for the Hamilton-Jacobi equation from the perspective of the invariant theory of Killing tensors

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Abstract: The theory of algebraic invariants of Killing tensors defined on Riemannian spaces of constant curvature under the action of the isometry group is described. The theory is illustrated by the computation of fundamental sets of invariants on three dimensional Euclidean space. The invariants are employed to classify the orthogonally separable coordinate webs for the Hamilton-Jacobi equation for the geodesics on $\mathbb{E}^3$ (joint work with Joshua Horwood and Roman Smirnov)

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1. Introduction

An orthogonally separable web on a n-dimensional pseudo-Riemannian manifold (M, g) is a set of n foliations of M by (n-1)-dimensional mutually orthogonal hypersurfaces with the property that the Hamilton-Jacobi equation for the geodesics

\[ g^{ij} \frac{\partial W}{\partial u^i} \frac{\partial W}{\partial u^j} = E \]  \hspace{1cm} (1)

has a complete integral of the form

\[ W(u, c) = \sum_{i=1}^{n} W_i(u^i, c) \]  \hspace{1cm} (2)

where \( u = (u^1, \ldots, u^n) \) is a coordinate system adapted to the web and

\( c = (c_1, \ldots, c_n) \) are the separation constants.

The coordinate system \( u \) is said to be adapted if the leaves of the foliation
are defined locally by

\[ u^i = \text{constant} \quad (3) \]

**Example:**

The elliptic-hyperbolic web on the Euclidean plane \( E_2 \)

Transformation from canonical Cartesian coords \((x, y)\) to adapted (separable) coords \((u, v)\):

\[ x = k \cosh u \cos v, \quad y = k \sinh u \sin v \quad (4) \]

The elliptic leaves are defined by

\[ u = \text{constant} \]

the hyperbolic leaves by

\[ v = \text{constant} \]
The constant \( k \) represents one-half the distance between the foci of the confocal families of ellipses and hyperbolas.

The Euclidean plane also admits the Cartesian, polar and parabolic webs which exhausts the possibilities in this case.

The theory of orthogonal separability of the HJ equation has been developed over more than one hundred years by Morera, Stäckel, Bôcher, Levi-Civita, Eisenhart, Kalnins & Miller, Benenti and many others. We use a geometric characterization of orthogonal separability due to Eisenhart as modified and extended by Benenti.
Theorem 1: A Hamiltonian $H = G + V$ is orthogonally separable iff there exists a valence two Killing tensor $K = K^{ij} \Omega^i \Omega^j$, with pointwise simple eigenvalues, orthogonally integrable eigenvectors and such that $d(\hat{K} \Omega V) = 0$, where the $(1,1)$ tensor $\hat{K}$ is given by $\hat{K} = Kg$. (A tensor satisfying the conditions of Theorem 1 is called a characteristic Killing tensor (CKT).)

2. Killing tensors

A valence $p$ Killing tensor on $(M, g)$ is a symmetric $(p,0)$ tensor the covariant components of which satisfy

$$\nabla_{(i_1, k_1, \ldots, i_{p+1})} K = 0$$

where $\nabla$ denotes the covariant derivative. Such tensors if they exist on $M$ define first integrals of the geodesic Hamiltonian
\[ H = \frac{1}{2} g_{ij} p_i p_j \]  

by

\[ K = K_i \ldots i_p p_i \ldots p_i \]  
in the sense that the Poisson bracket

\[ [K, H] = 0 \]  

iff (5) holds.

For \( p = 1 \)

\[ k = k^i p_i \]  
is a first integral iff \( k \) is a Killing vector which defines an infinitesimal isometry of \( M \). The condition (5) for \( p = 1 \) is equivalent to

\[ L_k g_{ij} = 0 \]  

where \( L \) denotes the Lie derivative operator.

For \( p = 2 \), we obtain a valence two Killing tensor which is very important in view of Theorem 1.
We now assume that $M$ is a $n$-dimensional space of constant curvature. By a result of Kalnins & Miller the separable coordinate webs for (1) are necessarily orthogonal. From the work of Delong, Takeuchi and Thompson any $(p,0)$ Killing tensor is expressible as a sum of symmetrized products of the Killing vectors of $M$ and the dimension $d$ of the vector space $K^p(M)$ of such tensors is given by

$$d = \frac{1}{n} \binom{n+p}{p+1} \binom{n+p-1}{p} \quad (11)$$

The isometry group $I(M)$ has dimension $\frac{n}{2} (n+1)$. By a recent result of McLennaghan, Milson & Smirnov its induced action on $K^p(M)$ defines a representation
of $I(M)$. We shall be particularly concerned with the functions $F: K^p \to \mathbb{R}$ that are invariant under the group $I(M)$ in the sense that

$$F(g.K) = F(K) \quad (12)$$

for all $K \in K^p$ and all $g \in I(M)$. Such $I(M)$-invariant functions are called Killing tensor invariants.

To obtain them we need the following results (Olver):

**Theorem 2.** Let $G$ be a Lie group acting regularly on a $n$-dimensional manifold $M$ with $m$-dimensional orbits. Then in a neighbourhood of each point $x_0 \in M$, there exist $n-m$ functionally independent $G$-invariants $\Delta_1, \ldots, \Delta_{n-m}$. Any other
G-invariant $F$ defined near $x_0$ can be locally expressed as an analytic function of the fundamental invariants by $F(\Delta_1, \ldots, \Delta_{n-m})$.

Proposition 1: Let $G$ be a connected Lie group of transformations acting regularly on $M$. A smooth real-valued function $F : M \to \mathbb{R}$ is $G$-invariant iff

$$V(F) = 0$$

for all $x \in M$ and for every infinitesimal generator $\nu$ of $G$.

In our application $G$ is the representation of $\text{I}(M)$ on $K^p$. The infinitesimal generators $\nu$ can be computed once the explicit form of the representation has been determined.
3. **Fundamental invariants of Killing tensors in $E_3$**

$p = 1$: Killing vectors

Let $(x, y, z) = (x_1, x_2, x_3)$ be a Cartesian coordinate system on $E_3$.

Isometry group $I(E_3)$:

$$x = \lambda \tilde{x} + \delta$$  \hspace{1cm} (13)

where $\lambda = (\lambda_{ij})$ is a constant orthogonal matrix and $\delta = (\delta_1, \delta_2, \delta_3)$ is a constant vector.

Infinitesimal generators (Lie algebra) of $I(E_3)$:

$$X_i = \frac{\partial}{\partial x_i}, \quad R_i = \varepsilon_{ijk} x_k x_j, \quad i, j, k = 1, 2, 3$$  \hspace{1cm} (14)

Commutator relations:

$$[X_i, X_j] = 0, \quad [X_i, R_j] = \varepsilon_{ijk} X_k$$

$$[R_i, R_j] = \varepsilon_{ijk} R_k$$  \hspace{1cm} (15)

Form of general Killing vector

$$V = a_i \ X_i + c_i \ R_i$$  \hspace{1cm} (16)

where $a_i, c_i, \ i = 1, 2, 3$ are arbitrary real numbers.
Transformation laws for the parameters:

In the tilde coordinate system

\[ V = \tilde{a}_i \tilde{x}_i + \tilde{c}_i \tilde{R}_i \]  

(17)

The transformation (13) induces the transformation

\[
\begin{align*}
\tilde{c}_i &= \lambda_{ji} c_j \\
\tilde{a}_i &= \lambda_{ji} a_j + \mu_{ij} c_j
\end{align*}
\]  

(18)

where

\[ \mu_{ij} = \epsilon_{ikl} \lambda_{lj} \delta_k \]  

(19)

The transformation defines a representation of $\mathfrak{I}(E_3)$ on $K^4$ (the adjoint representation).

Infinitesimal generators of (18):

\[
\begin{align*}
U_i &= \epsilon_{ijk} c_j \frac{\partial}{\partial a_k}, \quad i = 1, 2, 3 \\
V_i &= \epsilon_{ijk} \left( a_j \frac{\partial}{\partial a_k} + c_j \frac{\partial}{\partial a_k} \right)
\end{align*}
\]  

(20)

Commutator relations: same as (15)

$I(E_3)$ - invariants:

By Proposition 1 the invariants satisfy the pde's

\[ U_i (F) = 0, \quad V_i (F) = 0, \quad i = 1, 2, 3 \]  

(21)
Fundamental set of $\text{I}(E_3)$ invariants:

\[ \{ \Delta_1 := c_1^2 + c_2^2 + c_3^2, \quad \Delta_2 := a_1 c_1 + a_2 c_2 + a_3 c_3 \} \quad (22) \]

Obtained by solving (21) by the method of characteristics.

Invariant classification of Killing vectors in $E_3$:

<table>
<thead>
<tr>
<th>Classification</th>
<th>Invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>translational</td>
<td>$\Delta_1 = \Delta_2 = 0$</td>
</tr>
<tr>
<td>rotational</td>
<td>$\Delta_1 \neq 0, \Delta_2 = 0$</td>
</tr>
<tr>
<td>helicoidal</td>
<td>$\Delta_1 \neq 0, \Delta_2 \neq 0$</td>
</tr>
</tbody>
</table>

Canonical forms of Killing vectors in $E_3$:

<table>
<thead>
<tr>
<th>Classification</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>translational</td>
<td>$V = a_3 X_3$</td>
</tr>
<tr>
<td>rotational</td>
<td>$V = c_3 R_3$</td>
</tr>
<tr>
<td>helicoidal</td>
<td>$V = a_3 X_3 + c_3 R_3$</td>
</tr>
</tbody>
</table>

Obtained by means of (18).

Singular points of the Killing vector $V$ (16):

If $\Delta_2 = 0$ and $V \neq 0$, then $K$ has a line of singular points in the direction of $(c_1, c_2, c_3)$.

If $\Delta_2 \neq 0$, then $V$ has no singular points.
\( p = 2 \): valence two Killing tensors

Form of the general Killing tensor

\[ K = A_{ij} X_i \odot X_j + B_{ij} X_i \odot R_j + C_{ij} R_i \odot R_j \quad (23) \]

where \( A_{ij}, B_{ij}, \) \& \( C_{ij} \) are arbitrary real parameters which satisfy the following symmetry relations

\[ A_{ij} = A_{ji}, \quad C_{ij} = C_{ji} \quad (24) \]

Write

\[ A = [A_{ij}], \quad B = [B_{ij}], \quad C = [C_{ij}] \quad (25) \]

In terms of the natural basis

\[ K = K_{ij} X_i \odot X_j \quad (26) \]

where

\[ K_{ij} = A_{ij} + B_{ijk} E_{klm} x_l + C_{kl} E_{kmi} E_{lnj} x_m x_n \quad (27) \]

Only the differences \( b_{ii} - b_{ij} \) appear in \( K_{ij} \).

This is a reflection of the identity

\[ X_i \odot R_i = 0 \]
Transformation law for the parameters

The transformation (13) induces the transformation

\[ \tilde{A} = \lambda^t A \lambda + \frac{1}{2} (\lambda^t B \mu + \mu^t B^t \lambda) + \mu^t C \mu \] (28)

\[ \tilde{B} = \lambda^t B \lambda + 2 \lambda^t C \mu \] (29)

\[ \tilde{C} = \lambda^t C \lambda \] (30)

The above transformation defines a representation of SE(3) on \( K^2(\mathbb{R}^3) \).

Infinitesimal generators of (28)-(30)

They may be computed by taking derivatives in the directions of the translations and rotations at the identity.

Translation differential operators

\[ \tilde{U}_i = 2 \varepsilon_{ijkl} B_{kl} \frac{2}{2 A_{ik}} + \varepsilon_{jil} C_{kl} \frac{2}{2 B_{jk}}, i=1,2,3 \] (31)

Rotation differential operators

\[ \tilde{V}_i = (\varepsilon_{jil} A_{kl} + \varepsilon_{kl} A_{il}) \frac{2}{2 A_{ik}} + (\varepsilon_{jil} B_{kl} + \varepsilon_{kl} B_{il}) \frac{2}{2 B_{jk}} \]

\[ + (\varepsilon_{jil} C_{kl} + \varepsilon_{kl} C_{il}) \frac{2}{2 C_{jk}}, i=1,2,3 \] (32)

The coefficient matrix for the system has rank six almost everywhere.
\[
\begin{align*}
\Delta_1 &= B_{ii}, \\
\Delta_2 &= C_{ii}, \\
\Delta_3 &= B_{ij}C_{ij}, \\
\Delta_4 &= C_{ij}, \\
\Delta_5 &= B_{ij}B_{ji} + A_{ij}C_{ij}, \\
\Delta_6 &= B_{ij}C_{jk}C_{ki}, \\
\Delta_7 &= C_{ij}C_{jk}C_{ki}, \\
\Delta_8 &= C_{ij}[B_{jk}(B_{ik} + 2B_{ki}) + A_{jk}C_{ki}], \\
\Delta_9 &= \varepsilon_{ikm}\varepsilon_{jnk}B_{ij}B_{kl}B_{mn} - 2(B_{ij}B_{jk} + A_{ij}C_{ij})B_{kk} + 6B_{ij}A_{jk}C_{ki}, \\
\Delta_{10} &= B_{ij}(B_{ik}C_{kj} - 2B_{jk}C_{ki}) - (B_{ij}B_{ij} + A_{ij}C_{ij})C_{kk} + A_{ii}C_{ij}C_{kk}, \\
\Delta_{11} &= \varepsilon_{imn}\varepsilon_{jkn}B_{ij}B_{kl}C_{mn}C_{np} + B_{ij}[B_{ij}C_{kl}C_{kl} - C_{jk}(C_{kl}B_{il} + 4C_{t[k}B_{l]}i)] \\
&+ A_{ij}C_{ij}C_{kk}C_{kl}, \\
\Delta_{12} &= A_{ii}[(C_{ij}C_{kk} + 3C_{jk}C_{jk})C_{tt} - 4C_{jk}C_{kl}C_{el}] - 6A_{ij}C_{ij}C_{kl}C_{kl} \\
&+ 6B_{ij}[B_{ij}C_{kl}C_{kl} - C_{jk}[(B_{ik} - 2B_{ki})C_{tt} + 4C_{kk}B_{tt}]] \\
&+ 12\varepsilon_{imn}\varepsilon_{jkn}B_{ij}B_{kl}C_{mn}C_{np}, \\
\Delta_{13} &= A_{ij}(B_{ij}C_{kk}C_{el} + B_{jk}C_{kl}C_{el} - 2C_{ii}(B_{kk}C_{el})) \\
&+ A_{ii}C_{jk}(B_{jk}C_{tt} - B_{tt}C_{tt}) - B_{ij}[B_{ij}B_{kl}C_{kl} + 2C_{jk}B_{kk}B_{tt} \\
&+ B_{jk}B_{ik}C_{kl} - (B_{jk}B_{tt} + B_{ik}B_{el})C_{kl}], \\
\Delta_{14} &= 4A_{ii}A_{ij}C_{kk}C_{el} + 8A_{ij}(A_{jk}C_{kk}C_{el} + A_{kk}C_{[k]}C_{el]}i) + A_{ij}C_{ij}(A_{kk}C_{kl} \\
&+ 4B_{el}B_{el}) + 4C_{ij}B_{jk}A_{kk}B_{el} + 16A_{ij}C_{ij}B_{[k]l}B_{l}i, \\
\Delta_{15} &= A_{ij}C_{ij}[(C_{kk}C_{tt} - 3C_{kk}C_{tt})C_{mn} + 2C_{kl}C_{tn}C_{mn}] \\
&- 6A_{ij}C_{jk}C_{kl}C_{el}C_{mn} - 12C_{ij}B_{jk}(C_{kl}B_{[k]}C_{el}m + 2B_{el}C_{el}C_{mn})
\end{align*}
\]
4. Eigenvalue problem for valence two Killing tensors

Eigenvalue equation

$$K_{ij} X^i = \lambda g_{ij} X^j \quad (33)$$

or

$$K^{ij} X^j = \lambda X^i \quad (34)$$

where

$$K^{ij} = g^{ik} K_{kj} \quad (35)$$

Characteristic equation \((n = 3)\)

$$\lambda^3 - \text{tr} K \lambda^2 + \frac{1}{2} (\text{tr} K^2 - \text{tr} K^2) \lambda - \det K = 0 \quad (36)$$

Discriminant

$$D := A^2B^2 - 4B^3 - 27C^2 - 4A^3C + 18ABC \quad (37)$$

where

$$A = \text{tr} K, \quad B = \frac{1}{2} [(\text{tr} K)^2 - \text{tr} K^2], \quad C = \det K \quad (38)$$

Since \(K\) is symmetric \(D > 0\).

For distinct roots \(D > 0\).

\(D\) is a polynomial of degree six in \(x, y, z\).
Condition for orthogonally integrable eigenvectors

Tonolo–Schouten–Nijenhuis (TSN) conditions

\[ N^l_{ij; k} 9_k l = 0 \quad (39a) \]

\[ N^l_{ij; K j l} = 0 \quad (39b) \]

\[ N^l_{ij; K k} m K^m l = 0 \quad (39c) \]

where

\[ N^l_{ijk} = 4(K^i l K^j_{[i,k]} + K^j_{[i} K^k_{j]},l) \quad (40) \]

is the Nijenhuis tensor.

The conditions (39) are necessary and sufficient for the eigendirections of \( K \) to be orthogonally integrable when all the eigenvalues of \( K \) are distinct.

The TSN conditions yield ten quadratic, thirty-five cubic, and eighty-four quartic equations respectively in the Killing tensor parameters.

We have been unable to solve them directly to obtain the most general KT yielding OI eigenvectors.
5. Eisenhart's method in $E^3$

Eigenvalue equation for $K$

$$K_{i\ j\ \alpha} = \lambda_a g_{i\ j\ \alpha}, \quad (41)$$

where $i, j$ are natural basis indices and $\alpha$ a frame basis indices, $\lambda_a$ the $n$ simple, real eigenvalues of $K$, and the $h^j_\alpha$,

$a = 1, \ldots, n$ the corresponding eigenvectors normalized such that $g_{i\ j\ \alpha} h^i_\alpha h^j_\beta = \pm 1$.

Contract (41) with $h^i_\beta$:

$$K_{ab} = \lambda_a g_{ab} \quad (42)$$

where

$$g_{ab} = g_{i\ j\ \alpha} h^i_\alpha h^j_\beta = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n)$$

where $\varepsilon_i^2 = 1$, $i = 1, \ldots, n$.

The $h^i_\alpha$ define an orthonormal frame on $M$.

KT equations for $p = 2$:

$$\nabla_a K_{bc} = 0 \quad (43)$$

may be written as
\[ E_a \lambda_a = 0, \quad a = 1, \ldots, n \text{ (no sum)} \quad (44a) \]

\[ E_a E_b \lambda_a = 2 (\lambda_a - \lambda_b) \Gamma_{aba}, \quad a \neq b \quad (44b) \]

\[ (\lambda_a - \lambda_b) \Gamma_{cab} + (\lambda_b - \lambda_c) \Gamma_{abc} + (\lambda_c - \lambda_a) \Gamma_{bca} = 0, \quad a \neq b \neq c \neq a \quad (44c) \]

where \( \Gamma_{abc} \) denote the connection coefficients, &

\[ E_a = h^i_a \frac{2}{dx^i}, \quad E^a = h_i^a dx^i \quad (45) \]

Orthogonally integrable eigenvectors ⇒

\[ E^a = f_a dx^a, \quad E_a = (f_a)^{-1} \frac{2}{dx^a}, \quad a = 1, \ldots, n \text{ (no sum)} \quad (46) \]

Eqs (44) now read

\[ \frac{2 \lambda_a}{dx^a} = 0 \quad (47a) \]

\[ \frac{2 \lambda_a}{dx^b} = (\lambda_a - \lambda_b) \frac{2}{dx^c} (\log f_a^2) \quad (47b) \]

Integrability conditions

\[ \frac{\partial^2}{\partial x^a \partial x^b} (\log f_a^2) + \frac{2}{dx^a} (\log f_b^2) \frac{2}{dx^b} (\log f_a^2) = 0 \quad (48a) \]

\[ \frac{\partial^2}{\partial x^b \partial x^c} (\log f_a^2) - \frac{2}{dx^a} (\log f_b^2) \frac{2}{dx^c} (\log f_a^2) + \frac{2}{dx^b} (\log f_c^2) \frac{2}{dx^c} (\log f_a^2) \]

\[ + \frac{2}{dx^c} (\log f_b^2) \frac{2}{dx^b} (\log f_a^2) = 0 \quad (48b) \]

The solution of these equations in \( E^3 \) given

by Eisenhart (1934) are given below.
The eleven separable webs in $\mathbb{E}^3$

**Cartesian:** $(x, y, z)$

\[
d s^2 = dx^2 + dy^2 + dz^2 \\
x = x, \quad y = y, \quad z = z \\
-\infty < x, y, z < \infty \\
K_1^{ij} = \text{diag}(0, 1, 0) \\
K_2^{ij} = \text{diag}(0, 0, 1)
\]

**Circular cylindrical:** $(r, \theta, z)$

\[
d s^2 = dr^2 + r^2 d\theta^2 + dz^2 \\
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z \\
r \geq 0, \quad 0 \leq \theta < 2\pi, \quad -\infty < z < \infty \\
K_1^{ij} = \text{diag}(0, r^4, 0) \\
K_2^{ij} = \text{diag}(0, 0, 1)
\]

**Parabolic cylindrical:** $(\mu, \nu, z)$

\[
d s^2 = (\mu^2 + \nu^2)(d\mu^2 + d\nu^2) + dz^2 \\
x = \frac{1}{2}(\mu^2 - \nu^2), \quad y = \mu \nu, \quad z = z \\
\mu \geq 0, \quad -\infty < \nu < \infty, \quad -\infty < z < \infty \\
K_1^{ij} = \text{diag}(\nu^2 g_{11}, -\mu^2 g_{22}, 0) \\
K_2^{ij} = \text{diag}(0, 0, 1)
\]

**Elliptic-hyperbolic:** $(\eta, \psi, z)$

\[
d s^2 = a^2(\cosh^2 \eta - \cos^2 \psi)(d\eta^2 + d\psi^2) + dz^2 \\
x = a \cosh \eta \cos \psi, \quad y = a \sinh \eta \sin \psi, \quad z = z \\
\eta \geq 0, \quad 0 \leq \psi < 2\pi, \quad -\infty < z < \infty, \quad a > 0 \\
K_1^{ij} = \text{diag}(a^2 \cos^2 \psi g_{11}, a^2 \cosh^2 \eta g_{22}, 0) \\
K_2^{ij} = \text{diag}(0, 0, 1)
\]

**Spherical:** $(r, \theta, \phi)$

\[
d s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\
x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \\
r \geq 0, \quad 0 \leq \theta < \pi, \quad 0 \leq \phi < 2\pi \\
K_1^{ij} = \text{diag}(0, r^4, r^4 \sin^2 \theta) \\
K_2^{ij} = \text{diag}(0, 0, r^4 \sin^4 \theta)
\]

**Prolate spheroidal:** $(\eta, \theta, \psi)$

\[
d s^2 = a^2(\sinh^2 \eta + \sin^2 \theta)(d\eta^2 + d\theta^2) + a^2 \sinh^2 \eta \sin^2 \theta d\psi^2 \\
x = a \sinh \eta \sin \theta \cos \psi, \quad y = a \sinh \eta \sin \theta \sin \psi, \quad z = a \cosh \eta \cos \theta \\
\eta \geq 0, \quad 0 \leq \theta < \pi, \quad 0 \leq \psi < 2\pi, \quad a > 0 \\
K_1^{ij} = \text{diag}(-a^2 \sin^2 \theta g_{11}, a^2 \sinh^2 \eta g_{22}, a^2(\sinh^2 \eta - \sin^2 \theta)g_{33}) \\
K_2^{ij} = \text{diag}(0, 0, a^2 \sinh^2 \eta \sin^2 \theta g_{33})
\]
\[
\begin{align*}
\text{Oblate spheroidal:} & \quad \{ \begin{align*}
ds^2 &= a^2(\cosh^2 \eta - \sin^2 \theta)(d\eta^2 + d\theta^2) + a^2 \cosh^2 \eta \sin^2 \theta \, d\psi^2 \\
x &= a \cosh \eta \sin \theta \cos \psi, \quad y = a \cosh \eta \sin \theta \sin \psi, \quad z = a \sinh \eta \cos \theta \\
\eta &\geq 0, \quad 0 \leq \theta < \pi, \quad 0 \leq \psi < 2\pi, \quad a > 0 \\
K_1^{ij} &= \text{diag}(a^2 \sin^2 \theta g_{11}, a^2 \cosh^2 \eta g_{22}, a^2(\cosh^2 \eta + \sin^2 \theta) g_{33}) \\
K_2^{ij} &= \text{diag}(0, 0, a^2 \cosh^2 \eta \sin^2 \theta \, g_{33}) \\
\end{align*} \\
\text{Parabolic:} & \quad \{ \begin{align*}
\ds^2 &= (\mu^2 + \nu^2)(d\mu^2 + d\nu^2) + \mu^2 \nu^2 \, d\psi^2 \\
x &= \mu \nu \cos \psi, \quad y = \mu \nu \sin \psi, \quad z = \frac{1}{2}(\mu^2 - \nu^2) \\
\mu &\geq 0, \quad \nu \geq 0, \quad 0 \leq \psi < 2\pi \\
K_1^{ij} &= \text{diag}(-\nu^2 g_{11}, \mu^2 g_{22}, (\mu^2 - \nu^2) g_{33}) \\
K_2^{ij} &= \text{diag}(0, 0, \mu^2 \nu^2 g_{33}) \\
\end{align*} \\
\text{Conical:} & \quad \{ \begin{align*}
\ds^2 &= dr^2 + \frac{r^2(\theta^2 - \lambda^2)}{(\theta^2 - b^2)(c^2 - \lambda^2)} \, d\theta^2 + \frac{r^2(\theta^2 - \lambda^2)}{(b^2 - \lambda^2)(c^2 - \lambda^2)} \, d\lambda^2 \\
x^2 &= \frac{(r \theta \lambda)}{b c}, \quad y^2 = \frac{r^2(\theta^2 - b^2)(b^2 - \lambda^2)}{b^2(c^2 - b^2)}, \quad z^2 = \frac{r^2(c^2 - \theta^2)(c^2 - \lambda^2)}{c^2(c^2 - b^2)} \\
r &\geq 0, \quad b^2 < \theta^2 < c^2, \quad 0 < \lambda^2 < b^2, \\
K_1^{ij} &= \text{diag}(0, r^2 \lambda^2 g_{22}, r^2 \theta^2 g_{33}) \\
K_2^{ij} &= \text{diag}(0, r^2 g_{22}, r^2 g_{33}) \\
\end{align*} \\
\text{Paraboloidal:} & \quad \{ \begin{align*}
\ds^2 &= \frac{(\mu - \nu)(\mu - \lambda)}{(\mu - b)(\mu - c)} \, d\mu^2 + \frac{(\mu - \nu)(\lambda - \nu)}{(b - \nu)(c - \nu)} \, d\nu^2 \\
&\quad + \frac{\lambda - \mu}{(b - \lambda)(\lambda - c)} \, d\lambda^2 \\
x^2 &= \frac{(\mu - b)(\mu - \lambda)}{b - c}, \quad y^2 = \frac{4(\mu - c)(\mu - \nu)(\lambda - c)}{b - c}, \quad z = \mu + \nu + \lambda - b - c \\
\mu &< \nu < \lambda < b < \mu < \infty \\
K_1^{ij} &= \text{diag}(2(\nu + \lambda) g_{11}, 2(\lambda + \mu) g_{22}, 2(\mu + \nu) g_{33}) \\
K_2^{ij} &= \text{diag}(-4\nu \lambda g_{11}, -4\lambda \mu g_{22}, -4\mu \nu g_{33}) \\
\end{align*} \\
\text{Ellipsoidal:} & \quad \{ \begin{align*}
\ds^2 &= \frac{(\theta - \eta)(\theta - \lambda)}{4(a - \eta)(b - \eta)(c - \eta)} \, d\eta^2 + \frac{(\theta - \eta)(\theta - \lambda)}{4(a - \theta)(b - \theta)(c - \theta)} \, d\theta^2 \\
&\quad + \frac{4(a - \eta)(b - \lambda)(c - \lambda)}{(a - \eta)(a - \theta)(a - \lambda)} \, d\lambda^2 \\
x^2 &= \frac{(a - b)(a - c)}{a - \eta}, \quad y^2 = \frac{(b - \eta)(b - \theta)(b - \lambda)}{(b - a)(b - c)}, \quad z^2 = \frac{(c - \eta)(c - \theta)(c - \lambda)}{(c - a)(c - b)} \\
a > \eta > b > \theta > c > \lambda \\
K_1^{ij} &= \text{diag}(-\theta + \lambda) g_{11}, -(\lambda + \eta) g_{22}, -(\eta + \theta) g_{33}) \\
K_2^{ij} &= \text{diag}(\theta \lambda g_{11}, \lambda \eta g_{22}, \eta \theta g_{33}) \\
\end{align*} \}
\end{align*}
\]
1. Cartesian web

\[ K^{ij} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \]

2. Circular cylindrical web

\[ K^{ij} = \begin{pmatrix} a_1 + c_3 y^2 & -c_3 x y & 0 \\ -c_3 x y & a_1 + c_3 x^2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \]

3. Parabolic cylindrical web

\[ K^{ij} = \begin{pmatrix} a_1 & b_{23} y & 0 \\ b_{23} y & a_1 - 2b_{23} x & 0 \\ 0 & 0 & a_3 \end{pmatrix} \]

4. Elliptic-hyperbolic web

\[ K^{ij} = \begin{pmatrix} a_1 + c_3 y^2 & -c_3 x y & 0 \\ -c_3 x y & a_2 + c_3 x^2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad \frac{a_1 - a_2}{c_3} > 0 \]

5. Spherical web

\[ K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 x y & -c_2 x z \\ -c_3 x y & a_1 + c_3 x^2 + c_2 z^2 & -c_2 y z \\ -c_2 x z & -c_2 y z & a_1 + c_2 x^2 + c_2 y^2 \end{pmatrix} \]

6. Prolate spheroidal web

\[ K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 x y & -c_2 x z \\ -c_3 x y & a_1 + c_3 x^2 + c_2 z^2 & -c_2 y z \\ -c_2 x z & -c_2 y z & a_3 + c_2 x^2 + c_2 y^2 \end{pmatrix}, \quad \frac{a_3 - a_1}{c_2} > 0 \]

7. Oblate spheroidal web

\[ K^{ij} = \begin{pmatrix} a_1 + c_2 z^2 + c_3 y^2 & -c_3 x y & -c_2 x z \\ -c_3 x y & a_1 + c_3 x^2 + c_2 z^2 & -c_2 y z \\ -c_2 x z & -c_2 y z & a_3 + c_2 x^2 + c_2 y^2 \end{pmatrix}, \quad \frac{a_3 - a_1}{c_2} < 0 \]

8. Parabolic web

\[ K^{ij} = \begin{pmatrix} a_1 - 2b_{12} z + c_3 y^2 & -c_3 x y & b_{12} x \\ -c_3 x y & a_1 - 2b_{12} z + c_3 x^2 & b_{12} y \\ b_{12} x & b_{12} y & a_1 \end{pmatrix} \]
9. Conical web

$$K^{ij} = \begin{pmatrix}
    a_1 + c_2z^2 + c_3y^2 & -c_3xy & -c_2zx \\
    -c_3xy & a_1 + c_3x^2 + c_1z^2 & -c_1yz \\
    -c_2zx & -c_1yz & a_1 + c_1y^2 + c_2x^2
\end{pmatrix}$$

10. Paraboloidal web

$$K^{ij} = \begin{pmatrix}
    a_1 - 2b_{12}z + c_3y^2 & -c_3xy & b_{12}x \\
    -c_3xy & a_2 + 2b_{21}z + c_3x^2 & -b_{21}y \\
    b_{12}x & -b_{21}y & a_3
\end{pmatrix}$$

$$b_{12}[b_{12}b_{21} + c_3(a_2 - a_3)] + b_{21}[b_{12}b_{21} + c_3(a_1 - a_3)] = 0$$

11. Ellipsoidal web

$$K^{ij} = \begin{pmatrix}
    a_1 + c_2z^2 + c_3y^2 & -c_3xy & -c_2zx \\
    -c_3xy & a_2 + c_3x^2 + c_1z^2 & -c_1yz \\
    -c_2zx & -c_1yz & a_3 + c_1y^2 + c_2x^2
\end{pmatrix}$$

$$(a_1 - a_2)c_1c_2 + (a_2 - a_3)c_2c_3 + (a_3 - a_1)c_3c_1 = 0$$

**Notation**

$$A_{ij} = \begin{pmatrix}
    a_1 & a_3 & a_2 \\
    a_3 & a_2 & a_1 \\
    a_2 & a_1 & a_3
\end{pmatrix}$$

$$B_{ij} = \begin{pmatrix}
    b_{11} & b_{12} & b_{13} \\
    b_{21} & b_{22} & b_{23} \\
    b_{31} & b_{32} & b_{33}
\end{pmatrix}$$

$$C_{ij} = \begin{pmatrix}
    \gamma_1 & \gamma_3 & \gamma_2 \\
    \gamma_3 & \gamma_2 & \gamma_1 \\
    \gamma_2 & \gamma_1 & \gamma_3
\end{pmatrix}$$
6. **Web symmetry**

An examination of the separable webs in \( \mathbb{E}^2 \) and \( \mathbb{E}^3 \) leads to the following definition:

Let \( K \) denote a characteristic Killing tensor on \( (M,g) \). Let \( \phi_t \) denote a one parameter group of isometries. The separable web defined by \( K \) is said to be \( \phi_t \)-symmetric iff

\[
\phi_t \ast K = K. \tag{52}
\]

The infinitesimal version of is

\[
\mathcal{L}_V K = 0, \tag{53}
\]

where \( V \) is the infinitesimal generator of \( \phi_t \).

Examples in \( \mathbb{E}^3 \):

- Cartesian web,
- Cylindrical webs,
- Rotational webs.
Elliptic-hyperbolic cylindrical web
Oblate spheroidal web
Ellipsoidal web
7. **Invariant classification of the separable webs in $\mathbb{E}^3$**

The eleven separable webs can be divided into three groups according to the following table.

<table>
<thead>
<tr>
<th>Separable webs in $\mathbb{E}^3$</th>
<th>Translational</th>
<th>Rotational</th>
<th>Asymmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian</td>
<td>spherical</td>
<td>conical</td>
<td></td>
</tr>
<tr>
<td>circular cylindrical</td>
<td>paraboloidal</td>
<td></td>
<td></td>
</tr>
<tr>
<td>parabolic cylindrical</td>
<td>oblate spheroidal</td>
<td></td>
<td></td>
</tr>
<tr>
<td>elliptic cylindrical</td>
<td>ellipsoidal</td>
<td></td>
<td></td>
</tr>
<tr>
<td>parabolic</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

More precisely, we say that a separable web defined by $K$ is a translational (rotational) web if it is translational (rotational) symmetric, that is, there exists a translational (rotational) Killing vector $V$. 
such that

$$\mathcal{L}_V K = 0$$  \hspace{1cm} (54)

We observe that the canonical translation webs 1-4 admit a translational Killing vector

$$V = \frac{2}{2z}, \text{ while the rotational webs 5-8 admit a rotational Killing vector}$$

$$V = \alpha \frac{2}{\partial y} - y \frac{2}{\partial x}$$

We call the webs admitting the above symmetries the **symmetrical webs**.

We now proceed more generally to define the subspace of translational Killing tensors

$$K_T^2(E^3)$$ of $$K^2(E^3)$$ to be the subspace admitting the Killing vector $$\frac{2}{2z}$$ and
satisfying the TSN conditions. A general element of this subspace has the form

\[
K^T = \begin{pmatrix}
  a_1 + 2b_{13}y + c_3y^2 & a_3 - b_{13}x + b_{23}y - c_3xy & 0 \\
  a_3 - b_{13}x + b_{23}y - c_3xy & a_2 - 2b_{23}x + c_3x^2 & 0 \\
  0 & 0 & a_3
\end{pmatrix}
\]  
(55)

Observe that \( K^2_T(E^3) \) is invariant under \( SE(E^2) \) the isometry group, and that the upper 2\times2 block is the general Killing tensor in \( E^2 \).

**Fundamental set of invariants for \( K^2_T(E^3) \)**

\[
\Delta_1 = c_3, \quad \Delta_2 = \left[ b_{13} - b_{23} + 4c_3(a_2-a_1)^2 \right] + 4(b_{13}b_{23} - 4a_3c_3)^2
\]  
(56)

Define the subspace of rotational Killing tensors \( K^2_R(E^3) \) to be the subspace admitting the Killing vector

\[
x \frac{2}{2y} - y \frac{2}{2x}
\]

and satisfying the TSI conditions.
The general element of this subspace has the form

$$K_{ij}^R = \begin{pmatrix} a_1 - 2b_{12}z + c_2 z^2 + c_3 y^2 & -c_3 xy & b_{12}x - c_2 xz \\ -c_3 xy & a_1 - 2b_{12}z + c_3 x^2 + c_2 z^2 & b_{12}y - c_2 yz \\ b_{12}x - c_2 xz & b_{12}y - c_2 yz & a_3 + c_2 x^2 + c_3 y^2 \end{pmatrix}$$

(57)

Observe that $K_R^2(\Gamma^3)$ is invariant under rotations about the z-axis and contains all the canonical Killing tensors 5-8 as special cases.

**Fundamental set of invariants for $K_R^2(\Gamma^3)$**

$$\Delta_1 = C_2, \quad \Delta_2 = b_{12}^2 + c_2(a_3 - a_1), \quad \Delta_3 = a_3, \quad \Delta_4 = c_3$$

(59)

**Classification procedure for the orthogonally separable webs**

By Theorem 1 a valence two Killing tensor with distinct eigenvalues is orthogonally integrable
eigenvectors defines an orthogonally separable web in $E^3$. Such Killing tensors are called characteristic Killing tensors (CKT).

The classification procedure thus devolves to the classification of CKTs.

Let $K \in K^2(1E^3)$ be the KT to be classified.

If $K$ is constant ($b_{ij} = c_{ij} = 0$), then it characterizes the Cartesian web modulo a rotation.

Assume $K$ is not constant.

Step 1. Determine whether $K$ characterizes a translational, rotational or asymmetric web according to the type of Killing vector it admits.
Step 2. Use the invariant classification schemes given in the following three tables which classify the translational webs, the rotational webs, and the asymmetric webs.

**Table 1. Invariant classification of translational KTs in \( \mathbb{E}^3 \)**

<table>
<thead>
<tr>
<th>Coordinate web</th>
<th>Invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian</td>
<td>( \Delta_1 = 0, \Delta_2 = 0 )</td>
</tr>
<tr>
<td>circular cylindrical</td>
<td>( \Delta_1 \neq 0, \Delta_2 = 0 )</td>
</tr>
<tr>
<td>parabolic cylindrical</td>
<td>( \Delta_1 = 0, \Delta_2 \neq 0 )</td>
</tr>
<tr>
<td>elliptic-hyperbolic cylindrical</td>
<td>( \Delta_1 \neq 0, \Delta_2 \neq 0 )</td>
</tr>
</tbody>
</table>

**Table 2. Invariant classification of rotational KTs in \( \mathbb{E}^3 \)**

<table>
<thead>
<tr>
<th>Coordinate web</th>
<th>Invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>circular cylindrical</td>
<td>( \Delta_1 = 0, \Delta_2 = 0 )</td>
</tr>
<tr>
<td>spherical</td>
<td>( \Delta_1 \neq 0, \Delta_2 = 0 )</td>
</tr>
<tr>
<td>prolate spheroidal</td>
<td>( \Delta_1 \neq 0, \Delta_2 &gt; 0 )</td>
</tr>
<tr>
<td>oblate spheroidal</td>
<td>( \Delta_1 \neq 0, \Delta_2 &lt; 0 )</td>
</tr>
<tr>
<td>parabolic</td>
<td>( \Delta_1 = 0, \Delta_2 \neq 0 )</td>
</tr>
</tbody>
</table>
Table 3. Invariant classification of asymmetric KTs in $\mathbb{E}^3$

<table>
<thead>
<tr>
<th>Coordinate web</th>
<th>Invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>paraboloidal</td>
<td>$(\Sigma_1, \Sigma_2) = (0,0)$</td>
</tr>
<tr>
<td>ellipsoidal</td>
<td>$(\Sigma_1, \Sigma_2) \neq (0,0), \Sigma = 0$ OR $(\Sigma_1, \Sigma_2) \neq (0,0), \Sigma \neq 0, (\Sigma_4, \Sigma_5, \Sigma_6) \neq (0,0,0)$</td>
</tr>
<tr>
<td>conical</td>
<td>$(\Sigma_1, \Sigma_2) \neq (0,0), \Sigma_3 \neq 0, (\Sigma_4, \Sigma_5, \Sigma_6) = (0,0,0)$</td>
</tr>
</tbody>
</table>

Definitions:

\[
\begin{align*}
\Sigma_1 &= \Delta_2^2 - \Delta_4 \\
\Sigma_2 &= \Delta_2^3 - \Delta_7 \\
\Sigma_3 &= 3 \Delta_4 - \Delta_2^2 \\
\Sigma_4 &= \Delta_2 \Delta_5 - 3 \Delta_8 - 2 \Delta_{10} \\
\Sigma_5 &= \Delta_2 \Delta_{10} + \Delta_4 \Delta_5 - \Delta_{11} \\
\Sigma_6 &= \Delta_2 [2 \Delta_2 (10 \Delta_2 \Delta_5 + 24 \Delta_8 - 3 \Delta_{10}) \\
& \quad - 72 \Delta_{11} + \Delta_{12}] - 48 \Delta_4 \Delta_8 - 20 \Delta_5 \Delta_7 + 16 \Delta_{15}
\end{align*}
\]
To implement Step 1 of the procedure let

V be the general Killing tensor (66) and impose the condition

$$\mathcal{L}_V K = 0$$

This equation yields a linear system of equations in the six Killing vector parameters $a_i, c_i$ which may be readily solved.

The general solution has the form

$$V = l_1 V_1 + \ldots + l_n V_n,$$

where $n \leq 6$, $l_i$, $i=1,\ldots,n$ are arbitrary non-zero constants and \{V_1, \ldots, V_6\} is a linearly independent set of Killing vectors.

If $V=0$, is the only solution of (44), $K$ does not admit a Killing vector and hence characterizes an asymmetric web, which may be classified by Table 3.
Otherwise we classify each of the $V_i$ according to whether they are translational rotational or helicoidal. If one of the $V_i$ is translational, then $K$ characterizes a translational web, otherwise $K$ characterizes a rotational web. We next transform $V$ to its appropriate canonical form $X_3$ or $R_3$ by means of a translation or a rotation. We apply the same transformation to $K$ which places it either in $K_T^2(IE^3)$ or $K_R^2(IE^3)$. Finally, we classify the transformed $K$ using Table 1 or Table 2 as appropriate.
Example:

\[ K_{ij} = \begin{pmatrix} 2y + 2z & -x - y & -x - z \\ -x - y & 2x + 2z & -y - z \\ -x - z & -y - z & 2x + 2y \end{pmatrix} \]  

(68)

is compatible with the Calogero-Moser potential

\[ V = (x-y)^{-2} + (y-z)^{-2} + (z-x)^{-2} \]  

(69)

TSN conditions \( \Rightarrow \) (68) has orthogonally integrable eigenvectors.

\( D > 0 \Rightarrow \) (68) has distinct eigenvalues

\[ L \sqrt{K} = 0 \]

has the solution

\[ V = (y-z)X_1 + (z-x)X_2 + (x-y)X_3 \]  

(70)

\( \Delta_1 \neq 0, \Delta_2 = 0 \Rightarrow V \) is a rotational KV.
(70) can be transformed to the canonical form

\[ \tilde{V} = \frac{2}{2^2} \]

by the rotation

\[ X_{ij} = \frac{1}{16} \begin{pmatrix} 2 & 0 & \sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \\ -1 & -\sqrt{3} & \sqrt{2} \end{pmatrix} \] (72)

Applying the rotation to (68) yields

\[ \tilde{K} = \sqrt{3} \begin{pmatrix} 2 \tilde{x} & 0 & -\tilde{y} \\ 0 & 2 \tilde{z} & -\tilde{y} \\ -\tilde{x} & -\tilde{y} & 0 \end{pmatrix} \] (73)

From Table 2. \( \Delta_1 = 0, \Delta_2 \neq 0 \Rightarrow K \) characterizes a parabolic web.