CR-manifolds, Pseudo product structures and 2\textsuperscript{nd} order ODE

Gerd Schmalz
University of New England

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Introduction and Problem
Three geometric structures
Linearisation Problem from CR geometry

Solution
A priori information on automorphism
Shear invariant ODE

ODE/CR-manifolds with additional symmetries
### Outline

**Introduction and Problem**

**Solution**

**ODE/CR-manifolds with additional symmetries**

#### Three geometric structures

- Linearisation Problem from CR geometry

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<th>Second order ODE, complex holomorphic</th>
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<td>(y'' = B(x, y, y'))</td>
<td>2 hol. dir. fields (Z_1, Z_2) (\iff) 2 foliations by hol. curves</td>
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<td>non-involutivity ([Z_1, Z_2] \not\in \text{span}(Z_1, Z_2))</td>
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- CR-manifolds of dim 6, CR-dim=2

\[
\begin{align*}
D &= \text{span}(Z_1, Z_2) \\
J|_{Z_1} &= i, \quad J|_{Z_2} = -i \\
\text{special Levi form curvature condition}
\end{align*}
\]

\[
(x, y, p = \frac{dy}{dx})
\]

\[
\begin{align*}
Z_1 &= \frac{\partial}{\partial p} \\
Z_2 &= \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + B \frac{\partial}{\partial p}
\end{align*}
\]

- Embedding into \(\mathbb{C}^4\):

\[
\begin{align*}
\dot{x} &= 1, \quad \dot{y} = p, \\
\dot{p} &= B(x, y, p)
\end{align*}
\]

2\(^{nd}\) foliation encodes all structure

\[
\begin{align*}
\bar{w}_2 &= \phi(\bar{z}_2, z_1, w_1)
\end{align*}
\]
Mappings

(prolonged) point transformations

mappings that preserve $Z_1$ and $Z_2$ up to scale

CR-mappings
Most symmetric objects

\[ y'' = 0 \]

\[ \mathbb{F}(2, 1) \]

\[ \mathbb{C}P(2) \quad (\mathbb{C}P(2))^* \]

\[ y = cx + d \]

\[ \frac{w_1 - \bar{w}_2}{2i} = z_1\bar{z}_2 \]

(polarisation of \( \text{Im} w = |z^2| \))

acts by projective transformations

acts as polarisation of \( \text{SU}(2, 1) \)

induced action

\( \text{PSL}(3, \mathbb{C}) \)
Sphere $\text{Im } w = |z|^2$ can be characterised by the property that there exist non-trivial automorphisms $\Phi$ with $\Phi(0) = 0$ and $d\Phi(0) = \text{id}$, namely

$$
\begin{align*}
    z &\mapsto \frac{z + aw}{1 - 2i\bar{a}z - (r + i|a|^2)w} \\
    w &\mapsto \frac{w}{1 - 2i\bar{a}z - (r + i|a|^2)w}
\end{align*}
$$

Is the analogous statement true for (elliptic) CR manifolds of codimension 2? What symmetries can appear? Describe manifolds with symmetries.
Easy:

- for given $B \Rightarrow$ (infinitesimal) automorphisms
- for given (infinitesimal) automorphism $\Rightarrow B$

Infinitesimal automorphisms: $\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial p}$

with $\phi = \frac{\partial \eta}{\partial x} + p \left( \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x} \right) - p^2 \frac{\partial \xi}{\partial y}$.

Solve

$$\xi \frac{\partial B}{\partial x} + \eta \frac{\partial B}{\partial y} + \phi \frac{\partial B}{\partial p} + \left( 2 \frac{\partial \xi}{\partial x} + 3p \frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y} \right) B - \frac{\partial^2 \eta}{(\partial x)^2}$$

$$+ p \left( \frac{\partial^2 \xi}{(\partial x)^2} - 2 \frac{\partial \eta}{\partial x \partial y} \right) + p^2 \left( 2 \frac{\partial^2 \xi}{\partial x \partial y} - \frac{\partial \eta}{(\partial y)^2} \right) + p^3 \frac{\partial^2 \xi}{(\partial y)^2} = 0$$

Difficulty: Don’t know neither $B$ nor $\xi, \eta$. 
Power series methods, normal form (adapted to defining equation):

- such CR-manifold is torsion-free
- $\phi$ is (algebraic) deformation of

$$z_1 \mapsto \frac{z_1}{1 - 2i\bar{a}z_1}, \quad w_1 \mapsto \frac{w_1}{1 - 2i\bar{a}z_1}$$

$$z_2 \mapsto z_2 + aw_2, \quad w_2 \mapsto w_2$$

which corresponds to

$$x \mapsto x + ty, \quad y \mapsto y, \quad p \mapsto \frac{p}{1 + tp}.$$ 

Cartan geometry, normal form (adapted to symmetry):

- in normal coordinates $\phi$ is exactly as above
- hence, has the same topology (curve of fixed points $y = p = 0$)
Consequence

\[ y \frac{\partial B}{\partial x} - p^2 \frac{\partial B}{\partial p} + 3pB = 0 \]

Solution

\[ B(x, y, p) = F(y, x - \frac{x}{p})p^3 \]

Due to regularity,

\[ B(x, y, p) = \sum_{j=0}^{3} f_j(y)(y - px)^{3-j}p^j. \]

Question: Can they be equivalent to \( B = 0 \)??
Theorem (Ezhov, S., 2005)

There are local coordinates \(x, y, p\) such that \(B\) takes the reduced form

\[ B = f_0(y)(y - px)^3 + f_1(y)p(y - px)^2. \]

Two reduced forms are equivalent if and only if they are equivalent under

\[ x \mapsto \frac{c_1 x}{1 - cy}, \quad y \mapsto \frac{c_2 y}{1 - cy}, \]

i.e. the remaining freedom in coordinate choice consists of three complex parameters \(c_1, c_2, c\).
Idea of proof.

Case 1: $y \frac{\partial}{\partial x}$ is the only shear-symmetry. Then preserving reduced form requires preserving $y \frac{\partial}{\partial x}$. Such mappings satisfy a pair of second order ODE ($\Rightarrow 4$ parameters).
But one parameter corresponds to the one-parametric shear symmetry group and

$$x \mapsto \frac{c_1 x}{1 - cy}, \quad y \mapsto \frac{c_2 y}{1 - cy}$$

is known to preserve the reduced form.

Case 2: There is a second shear symmetry. $\Rightarrow$ Study ODE with more symmetries.
Which of those ODE/CR-manifolds have additional automorphisms?

S. Lie classified ODE by (infinitesimal) symmetries

8 symmetries $\Rightarrow y'' = 0$

3 symmetries $\Rightarrow$ short list of ODE

2 symmetries $\Rightarrow y'' = f(y')$ (for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$)

or $y'' = \frac{f(y')}{x}$ (for $\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$)

1 symmetry $\Rightarrow y'' = f(x, y')$. 
Theorem (Ezhov, S.)

\[ y'' = 0 \text{ and } y'' = (y - xy')^3 \] are (up to equivalence) the only ODE with more than one shear symmetry.

\[ y'' = (y - xy')^3 \] is \( \text{SL}(2, \mathbb{C}) \) invariant.

The ODE with exactly two symmetries with fixed point 0 are equivalent to

\[ y'' = y^k(y - xy')^3 \quad \text{or} \quad y'' = y^\ell y'(y - xy')^2 + Cy^{2\ell+2}(y - xy')^3. \]

The additional automorphisms are

\[ (k + 2)x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} \quad \text{resp.} \quad (\ell + 2)x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}. \]
The corresponding CR manifolds

\[ w_1 + w_1^2 \bar{z}_2^2 - (\bar{w}_2 - z_1 \bar{z}_2)^2 = 0 \]

\[ \bar{w}_2 = z_1 \bar{z}_2 + \sqrt{k + 2 w_1 \bar{z}_2} \int \frac{dy}{y^2 \sqrt{1 + w_1^2 \bar{z}_2^k + 2}} \]

\[ \int \frac{dy}{y^2 \sqrt{1 + w_1^2 \bar{z}_2^k + 2}} \]

is a hypergeometric function, which satisfies non-linear ODE \( \Rightarrow \) (apparently new) relation
Shear invariant ODE with a second transitive symmetry:
The examples from above shifted in $x$-direction (by power series methods, comparison of parameters)
But corresponding coordinate transformations is highly transcendental.
Example: $f_0(y) \equiv 0$ and $f_1(y)$ satisfies

$$\left( \frac{f_1}{3f_1 + yf'_1} \right)'' = -2f_1.$$