The Divergent Beam X-Ray Transform

\[ D_z f(\omega) = \int_0^\infty f(z + t\omega) \, dt, \quad z \in \mathbb{R}^n, \ \omega \in S^{n-1}. \]

Parametrization in 2D:

\[ D_z f(\omega) = D f(\beta, \alpha), \quad z = (r \cos \beta, r \sin \beta) \]
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Symmetry:

\[ D f(\beta, \alpha) = D f(\beta + 2\alpha + \pi, -\alpha). \]
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The goal of tomography

The goal is to reconstruct $f(x)$ from finitely many measurements of line integrals $D f(\beta, \alpha)$. 
"In practice one can make only a finite number of measurements ... and the question which arises is how many observations should be made, and how should they be related to each other in order to reconstruct the object."

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We will use Shannon Sampling Theory to address this question.
Fundamental Question

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The answer depends on the frequency content of the data function $Df$, that is both on the size and the shape of the support of the Fourier transform of $Df$. 
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Change of variables:

$$g(s, t) = Df(\beta, \alpha), \ (s, t) \in [0, 1)^2 = \mathbb{T}^2.$$
The essential support of $\hat{g}$

A-priori information used: $f$ has support in unit disk and 'essential bandwidth' $b = 100$. 
The data is a function $g$ on some group $G$, here $G = T^2$. We measure $g$ on a discrete subgroup (lattice) $L$. 
Sampling lattices

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Characterization of sampling lattices

$P = \text{number of equidistant source positions}$
$Q = \text{number of equidistant rays measured for each source position}$
$N = \text{parameter for shift in fan of rays}$
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$L = \{(s_j, t_{jl}) : s_j = j \Delta s, \Delta s = 1/P, t_{jl} = l \Delta t + \delta_j, \Delta t = 1/Q, \delta_j = jN/(PQ) \}$

$j = 0, \ldots, P - 1, l = 0, \ldots, Q - 1$. 
Let \( g(x) \) be such that \( \hat{g}(\xi) = 0 \) for \( \xi \not\in K = [-b, b) \).
Let $g(x)$ be such that $\hat{g}(\xi) = 0$ for $\xi \notin K = [-b, b)$. If $0 < h \leq \pi/b$ then

$$g(x) = \frac{hb}{\pi} \sum_{l=-\infty}^{\infty} g(hl) \text{sinc}(b(x - hl)),$$

where $\text{sinc}(x) = (\sin x)/x$. 
Classical Sampling Theorem on $\mathbb{R}$

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where \( \text{sinc}(x) = (\sin x)/x \).

Note:
- \( g(x) \) is sampled on a subgroup \( L = h\mathbb{Z} \) of \( \mathbb{R} \).
- \( 2b \text{sinc}(bx) \) is the inverse Fourier transform of the indicator function \( \chi_K(\xi) \) of \( K \).
  \[\chi_K(\xi) = 1 \text{ for } \xi \in K \text{ and zero otherwise.}\]
Motivation of the Sampling Theorem

Assume $\hat{g}$ is very small outside a set $K$. 
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Assume $\hat{g}$ is very small outside a set $K$. Discrete Fourier Transform of measured data:

$$\text{DFT}(g) = c \sum_{y \in L} g(y) e^{-2\pi i \langle y, \xi \rangle}$$

$$= \hat{g}(\xi) + \sum_{0 \neq \eta \in L^\perp} \hat{g}(\xi + \eta)$$
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Hence \( \text{DFT}(g) \approx \hat{g}(\xi) \) if \( \xi \in K \) and \( \xi + \eta \notin K \), that is, if the translates \( K + \eta, \eta \in \mathbb{L}^\perp \) are disjoint.
Interpolation

\[ Sg(x) = \text{IFT} \left[ \chi_K \text{DFT}(g) \right] \]

\[ = c \sum_{y \in L} \tilde{\chi}_K(x - y) g(y) \]

\[ \chi_K = \text{indicator fct. of } K \]

\[ \tilde{\chi}_K = \text{IFT} [\chi_K] \]
Theorem 1 \( \text{Let } L = L(N, P, Q) \text{ a sampling lattice such that } K + \eta, \eta \in L^\perp \text{ are disjoint. For } z \in \mathbb{T}^2 \text{ define} \)

\[
Sg(z) = \frac{1}{PQ} \sum_{y \in L} \tilde{\chi}_K(z - y)g(y).
\]

Then

\[
|g(z) - Sg(z)| \leq 2 \int_{\mathbb{Z}^2 \setminus K} |\hat{g}(\zeta)| \, d\zeta.
\]
Achieving Efficiency

1. Find lattices $L$ as sparse as possible so that the translates $K + \eta$, $\eta \in L^\perp$ are disjoint.

2. Exploit the symmetry relation

$$Df(\beta, \alpha) = Df(\beta + 2\alpha + \pi, -\alpha).$$

This requires sampling theorems for sampling sets which are not lattices.
Choice of lattices

Find lattices $L$ as sparse as possible so that the translates $K + \eta, \eta \in L^\perp$ are disjoint.

Example: Standard lattice (N=0).

\[
L_S(P, Q) = \{(j/P, l/Q) : j = 0, \ldots, P - 1, l = 0, \ldots, Q - 1\}
\]

\[
L_S^\perp = \{(Pk, Qm) : k, m \in \mathbb{Z}\}
\]
Translates \( K + \eta, \eta \in L^\perp_S \)

\[
N = 0, \quad P = 156, \quad Q = 600, \quad |L_S| = PQ = 93,600
\]
More efficient lattice

\[ N = 110, \quad P = 330, \quad Q = 200 \quad |L| = PQ = 66,000 \]
Reconstruction

Two basic strategies for reconstructing the function $f$ from samples of $Df$.

1. **Direct.** Reconstruct directly from the sampled data. (This is the only possibility for local tomography.)
   Need for error analysis of reconstruction algorithm.

2. **Interpolated.** First use sampling theorem to interpolate data onto a denser grid. Then reconstruct from the interpolated data.
Direct local reconstruction of $\Lambda f$


Parallel-beam. (F. & Ritman, 2000)
Direct vs. interpolated for fan-beam

\[ f(x) = \left(1 - 100|x - x_0|^2\right)^3, \quad x_0 = (0.4, 0.7) \]

Theoretical explanation: Izen (2005)
Artifacts from undersampling

Left: Standard     Right: Efficient.
Top: $P$ too small
Bottom: $Q$ too large (!)
Undersampling by sampling more data

\[ N = 110, P = 330, Q = 240 > 200. \]
Extensions of the Sampling Theorem

Goal: Extend to sampling sets which are not subgroups but still retain group structure.

Periodic sampling: $S = \bigcup_{n=1}^{m}(x_n + L)$
Sampling set is invariant with respect to shifts by elements of $L$. (Well understood; see, e.g., Kohlenberg (1954), F. (1994), Izen (2005) and many other authors.)
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- Of course, these are not the only extensions! (See, e.g., Aldroubi & Unser, Feichtinger & Gröchenig, Zayed, ...)

Tomography and Sampling Theory – p. 19/34
Applications of periodic sampling in CT

- Additional efficient 2D sampling schemes
- "Preferred pitch" in 3D helical CT
- Higher resolution in 2D fan-beam CT

Applications of non-periodic sampling

- Higher resolution in 2D fan-beam CT
Exploiting Symmetry

\[ Df(\beta, \alpha) = Df(\beta + 2\alpha + \pi, -\alpha). \]

Standard lattice with constant detector shift:

\[ L_S = \{ (\beta_j, \alpha_l) : \beta_j = \frac{2\pi j}{P}, \quad \alpha_l = \frac{\pi(l + \delta)}{Q} \}, \]

\[ j = 0, \ldots, P - 1, \quad l = -Q/2, \ldots, Q/2 - 1, \quad \delta \geq 0 \}

‘Reflected lattice’

\[ L_R = \{ (\beta_j + 2\alpha_l + \pi, -\alpha_l) : (\beta_j, \alpha_l) \in L_S \}. \]

\( L_S \) and \( L_R \) are cosets of two different subgroups.
Key observation by Izen et al. (2005)

\( L_S \cup L_R \) is a union of \( Q/\gcd(P, Q/2) \) shifted copies of the smaller lattice

\[
L_P = \{(2\pi j/P, \pi l/\gcd(P, Q/2))
\mid j = 0, \ldots, P - 1, \ |l| \leq \gcd(P, Q/2)\}
\]

So the periodic sampling theorem can be applied!

(Izen, Rohler & Sastry (2005) used an alternative reconstruction method.)

Thus effective bandwidth \( b \) can be doubled by only having to double \( P \) but not \( Q \).
Use of periodic sampling

Mitchell (2005) used the periodic sampling theorem to incorporate the reflected data.
Standard (Direct) Reconstruction

Direct standard reconstruction

b=200, P=312, Q/2=q=300, alphamax=\pi/2
High-Resolution with periodic sampling

Periodic Sampling with 2 cosets

P=624, q=312, delta=0.25, R=3, b=400, rho=1, tau=0.95, k=624, nray=3
Gratton (2005) found an ingenious way to show that a non-periodic sampling theorem can be applied to the sampling set $L_S \cup L_R$ to exploit the symmetry.
Standard Rec. of Calibration Object

Real data.
High-Resolution Reconstruction

Challenge: Increased noise.
Possible remedy: Edge preserving denoising with TV-based algorithm (R. Hass, 2005).
Denoised Standard vs. Denoised High-R.
Standard vs. Denoised High-Resolution

![Standard](image1)

![Denoised High Resolution](image2)

![Graph](image3)
Three dimensions

Rich in potential sampling geometries. (Family of lines has 4 parameters. Need only a 3 parameter subfamily).

Investigation of sampling issues has begun (see, e.g, Desbat and Grangeat (2004), Gratton (2005)) but very much remains to be done.
Conclusions

Analysis of sampling requirements is a good idea. It helps understand and avoid artifacts and to increase resolution.
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- Both size and shape of the bandregion (spectrum) of $g$ matter.

- Awareness and proper use of a-priori information is important. Here we only assumed that $f$ has support in the unit disk and 'essential bandwidth' $b$. 
Conclusions, continued

Reconstruction can be done directly from the measured data or after interpolating data on a denser grid with the sampling theorem. The latter is often better.
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- Theory strictly applies only to sufficiently smooth functions, but turns out to give valuable guidance for other functions as well.

- Extensions of the classical sampling theorem allow to exploit the symmetry relation and also to construct additional efficient sampling schemes.

- Noise is a challenge for efficient sampling. Post-processing with edge preserving denoisers is promising.