The Equalized Net Diffusion (END) for PDE-based Image Restoration

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ABSTRACT

ISSUE: Most PDE-based image restoration models tend to either converging to a piecewise constant image or introducing undesired dissipation, which may cause a significant loss of fine structures.

CHALLENGE: Develop novel methods which can preserve not only fine structures but also slow transitions.

OBJECTIVES in THIS PRESENTATION:

- Analyze sources of the undesired excessive dissipation.
- Introduce a diffusion modulator, called the Equalized Net Diffusion (END) which can suppress the undesired dissipation.

PDE-BASED IMAGE RESTORATION

- **A common noise model**: for the observed u_0,

$$u_0 = u + g(u)\, v \tag{1}$$

where u is the desired image and $g(u)v$ denotes the noise with v having a zero mean.

- **A common restoration technique**:

$$u = \arg\min_u \left\{ \int_\Omega \rho(|\nabla u|)\, d\mathbf{x} + \frac{\lambda}{2} \int_\Omega \left(\frac{u_0 - u}{g(u)} \right)^2 d\mathbf{x} \right\}, \tag{2}$$

where ρ is an increasing function (often, convex) and $\lambda \geq 0$ denotes the constraint parameter.

- **The associated Euler-Lagrange equation**:

$$-\nabla \cdot \left(\rho'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) = \lambda (u_0 - u) \frac{g(u) + (u_0 - u) g'(u)}{g(u)^3}. \tag{3}$$

Examples

- **The total variation (TV) model:** $\rho(x) = x$ **and** $g(x) \equiv 1$ **(Rudin-Osher-Fatemi, 1992)**

$$\frac{\partial u}{\partial t} - \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = \lambda(u_0 - u).$$

- **The Perona-Malik (PM) model (1990):**
$\rho(x) = \frac{1}{2} K^2 \ln(1 + x^2/K^2)$ **and** $\lambda = 0$**:**

$$\frac{\partial u}{\partial t} - \nabla \cdot (c(|\nabla u|) \nabla u) = 0,$$

where $c(x) = \rho'(x)/x = (1 + x^2/K^2)^{-1}$**.**

- **Improved TV (ITV) models (Marquina-Osher, 2000):**

$$\frac{\partial u}{\partial t} - |\nabla u| \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = \lambda |\nabla u| (u_0 - u).$$

Examples (2)

- **A speckle-constrained TV model:** $g(x) = \sqrt{x}$
  (Krissian-Kikinis-Westin-Vosburgh, 2005)

$$\frac{\partial u}{\partial t} - \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = \lambda \, (u_0 - u) \, \frac{u_0 + u}{u^2}$$

- **Convex-Concave Anisotropic Diffusion (CCAD) Model:** $\rho(x) = x^{2-q}$**,** $q \geq 1$**, and** $g(u) = 1$
  (Kim-Lim, 2005)

$$\frac{\partial u}{\partial t} - |\nabla u|^q \, \nabla \cdot \left(\frac{\nabla u}{|\nabla u|^q} \right) = \lambda \, |\nabla u|^q \, (u_0 - u), \quad q \geq 1.$$

  Note: It is closely related to the PM model when $q = 2$**. However, the CCAD model introduces no staircasing.**

Numerical Performances

Staircasing (TV) & Dissipation (ITV)

SOURCES OF UNDESIRED DISSIPATION

Consider the TV model:

$$\frac{\partial u}{\partial t} - \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = \lambda \, (u_0 - u).$$

Let $v = u_0 - u$**, the residual.**

Then the associated <u>residual equation</u> becomes

$$\frac{\partial v}{\partial t} + \lambda \, v \; = \; -\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right)$$

<u>Remark</u>: Although $v(t=0) = 0$**, the residual at** $t > 0$ **becomes positive or negative at pixels where the image is concave or convex, respectively.**

In General,

We express the above PDE restoration models as

$$\frac{\partial u}{\partial t} - \mathcal{S}u = \mathcal{Q}(u_0 - u) \tag{4}$$

Then, the associated residual equation reads

$$\frac{\partial v}{\partial t} + \mathcal{Q}\,v = -\mathcal{S}u \tag{5}$$

Conclusion:

- The bigger the diffusion magnitude $|\mathcal{S}u|$ is, the larger dissipation occurs.
- It is a major source of undesired dissipation of conventional PDE-based restoration models.

DIFFUSION MODULATION

In order to overcome the drawback of (4), the model can be modulated by a function of diffusion operator itself:

$$\frac{\partial u}{\partial t} - F(\mathcal{S}u)\,\mathcal{S}u = R(u_0 - u), \tag{6}$$

where F is a positive function and $R = F(\mathcal{S}u)\,\mathcal{Q}$, the constraint term. Its residual equation reads

$$\frac{\partial v}{\partial t} + Rv = -F(\mathcal{S}u)\,\mathcal{S}u$$

Let $N(s) = F(s)s$, the net diffusion function.
Then, desired properties for the N are:

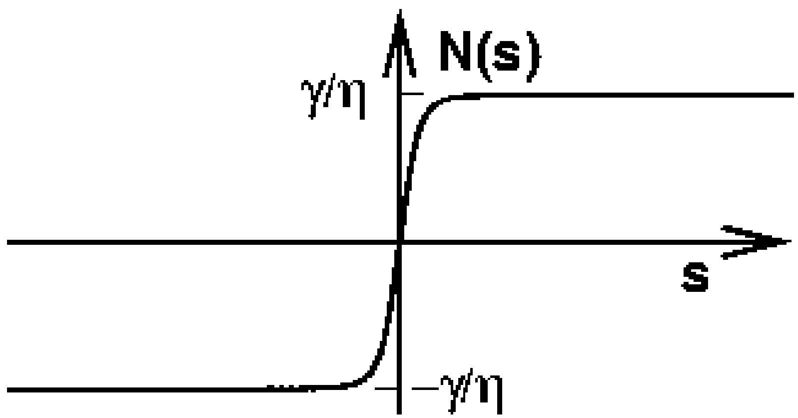
- increasing, origin-symmetric, &
- of little variation over a wide range of $|s| \geq s_0 > 0$

Equalized Net Diffusion (END)

An effective net diffusion function can be found as

$$N(s) = \frac{\gamma}{1 + \eta\,|s|}\,s,$$

for some positive constants γ and η.



Then, the corresponding END-incorporated model for the basic model (4) becomes

$$\frac{\partial u}{\partial t} - \frac{\gamma}{1 + \eta\,|\mathcal{S}u|}\mathcal{S}u = R\,(u_0 - u) \qquad (7)$$

NUMERICAL EXAMPLES

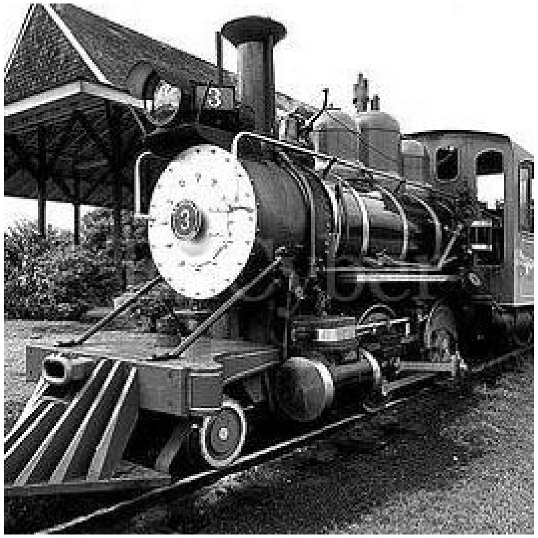
- **Time-stepping procedures:**
  - **A Crank-Nicolson, linearized for nonlinear terms**
  - **The alternating direction implicit (ADI) method**
- **The END parameters (γ and η): statistically,**
  - $\eta = 2.82 \cdot S_0^{-2.46}$**, where** S_0 **is an average of** $|\mathcal{S}u_0|$
  - **At each time level,** γ^n **is chosen such that the average of** $\gamma/(1+\eta|s^n|)$ **is one, i.e.,**

$$\gamma^n = N_x N_y \Big(\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \frac{1}{1+\eta\,|s_{ij}^n|} \Big)^{-1}$$

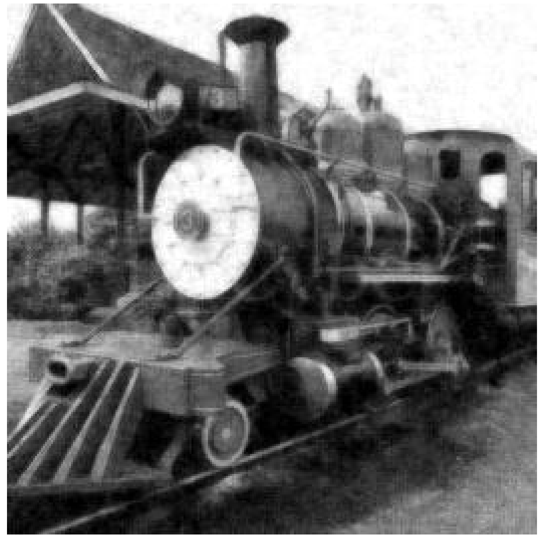
- **END-incorporated models converge in 3-5 iterations**

Train

(Original)

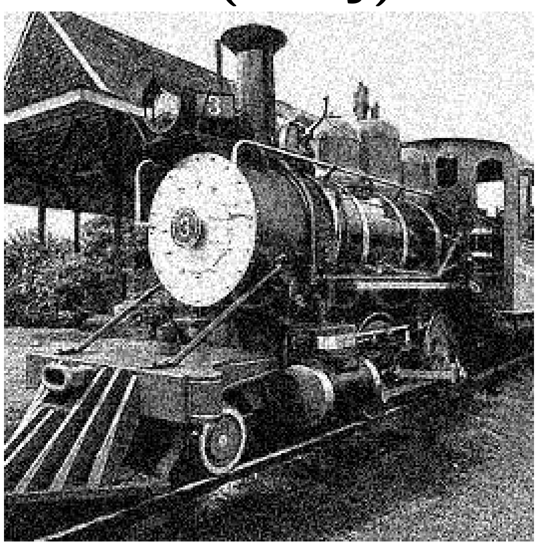


(ITV)

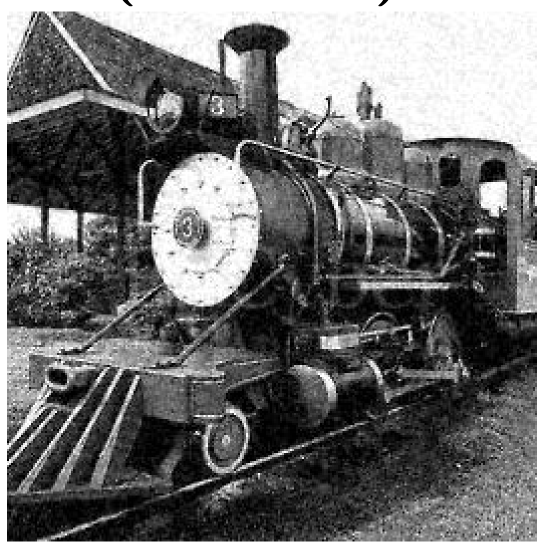


(CCAD)

(Noisy)



(END-ITV)



(END-CCAD)

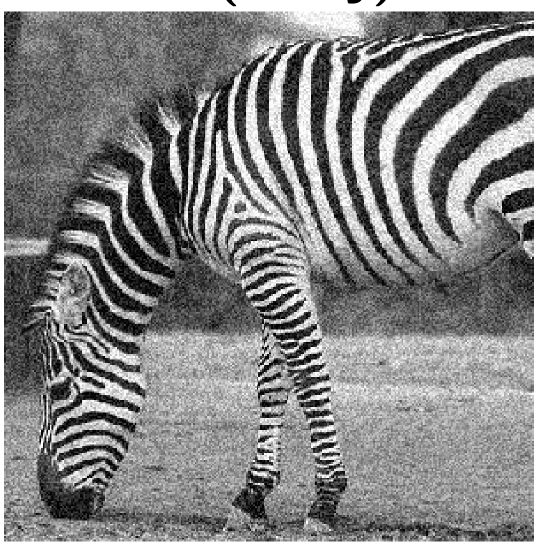
Zebra

(Original)

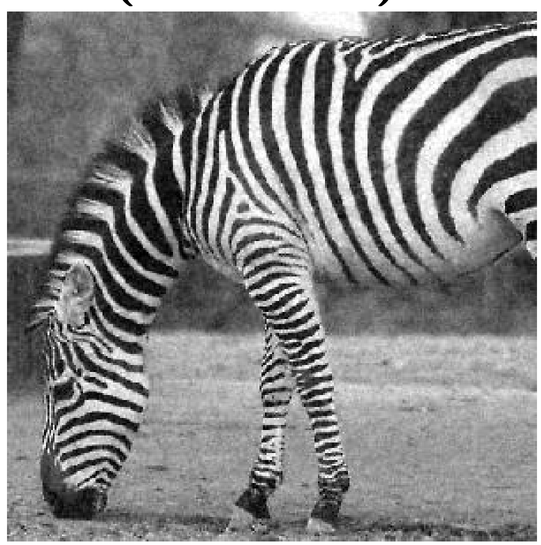
(ITV)

(CCAD)

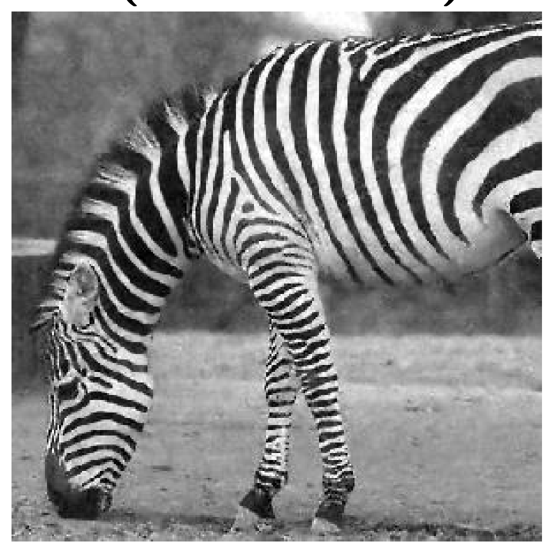
(Noisy)



(END-ITV)

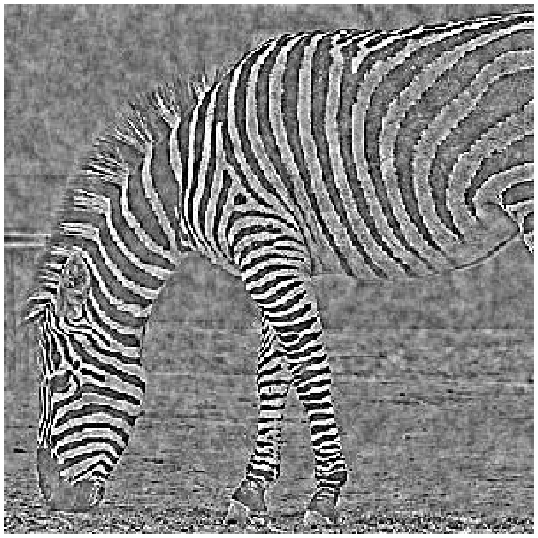


(END-CCAD)



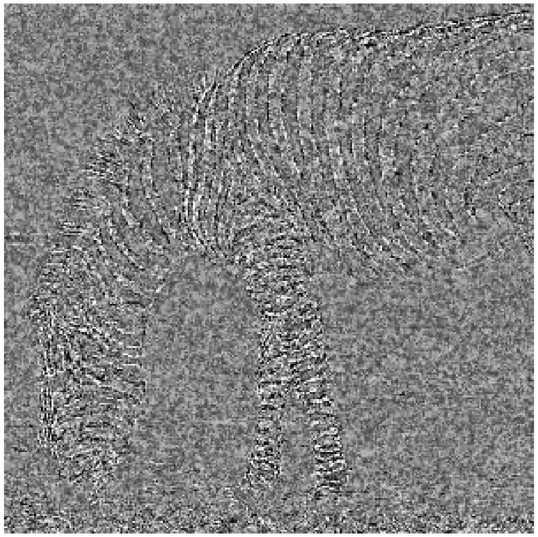
Magnified misfit: 3*(Original-Recovered)+128

(ITV)

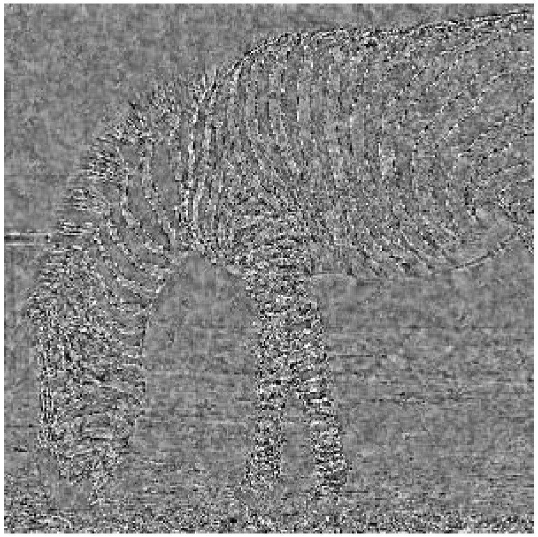


(CCAD)

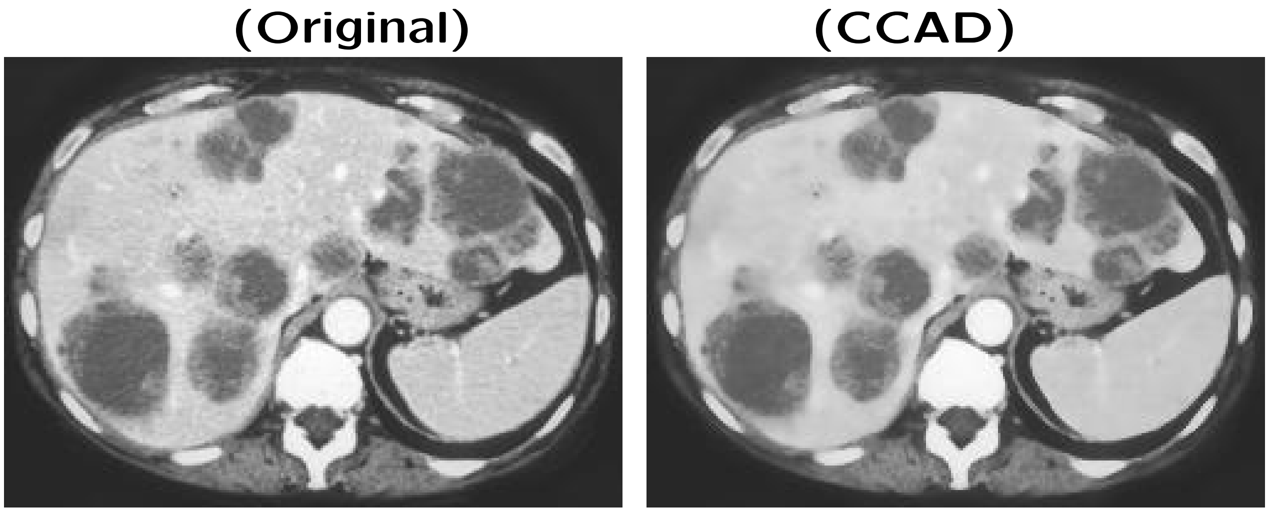
(END-ITV)

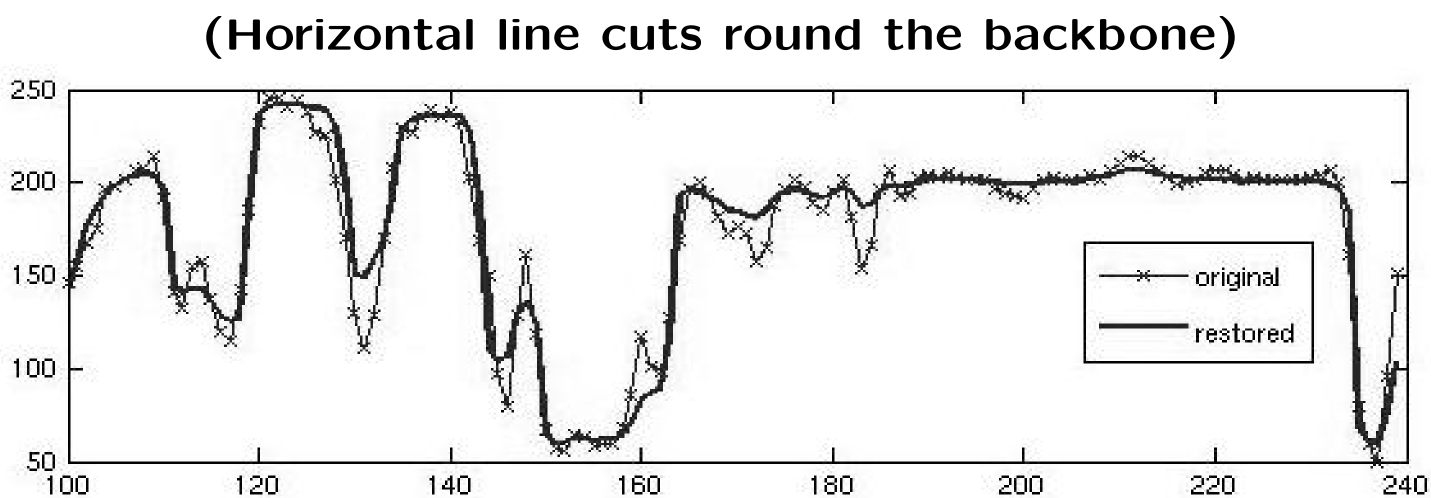


(END-CCAD)



CT Image: Liver





CONCLUSIONS

PDE-based Restoration Models:

- They tend to introducing undesired dissipation.
- The bigger the diffusion magnitude is, the larger dissipation occurs, which may cause a significant loss of fine structures.

Equalized Net Diffusion (END):

- END can restore not only fine structures but also slow transitions.
- END is applicable for various PDE-based models.

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