Mumford-Shah with A-Priori Medial-Axis Information

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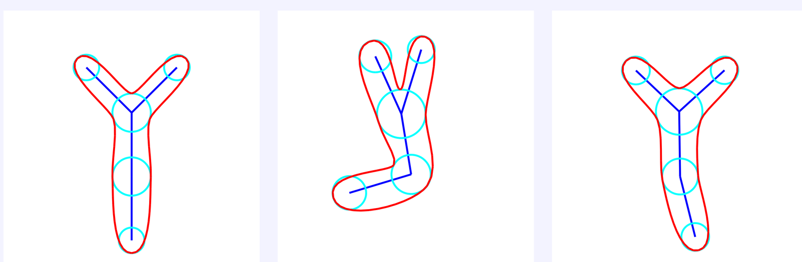
Abstract

We minimize the Mumford-Shah functional over a space of parametric shapes. In addition we penalize large deviations from a mean shape prior. This mean shape is the average of shapes obtained by segmenting a set of training images. The parametric description of the shapes is motivated by their medial axis representation.

The Shape Model



Three instances of the shape model.

Every instance of the shape model is represented by

1. the *skeleton* (dark blue lines) and
2. the *atoms* (light blue circles).

The shape is then obtained by fitting a spline curve (red) to the atoms. The *topology* of the skeleton is the same for every instance of the model. Thus, we have to provide the following information to describe an instance of the shape model:

1. *position* and *orientation* of a local coordinate system,
2. *length* and *direction* of the vectors corresponding to the skeleton edges,
3. *radii* of the atoms in the vertices of the skeleton.

For convenience we additionally specify a *global scaling* of the shape. All measures (the length of the vectors and the atom radii) are relative to this scaling.
This means that every shape is an element of the following product space M:

$$M = \underbrace{\mathbb{R}^+}_{\text{scaling}} \times \underbrace{\mathbb{R}^2 \times S^1}_{\text{coordinate system}} \times \underbrace{\left(\mathbb{R}^+ \times S^1\right)^m}_{\text{edge vectors}} \times \underbrace{\left(\mathbb{R}^+\right)^n}_{\text{atom radii}} .$$

The shape manifold M.

Here m denotes the number of edges of the skeleton and n the number of atoms, i.e. the number of vertices of the skeleton. In the above example $m = 4$ and $n = 5$. Note that M is a product of Lie groups and hence a Lie group itself. We call M the *shape manifold*. In particular M is a Riemannian manifold with a metric, which is invariant under the the group action.

The Simplified Mumford-Shah Functional

Assume an image $f : \Omega \to \mathbb{R}$. The following simplified functional [1] is motivated by the original Mumford-Shah functional [5, 6]. Assume that $C : [0,1] \to \Omega$ is a differentiable Jordan curve. Denote the area inside C as $\mathcal{I}(C)$ and the area outside as $\mathcal{O}(C)$. Then

$$I(C) = \int_{\mathcal{I}(C)} \left(u_1(C) - f\right)^2 dx + \int_{\mathcal{O}(C)} \left(u_2(C) - f\right)^2 dx \ ,$$

where

$$u_1(C) = \frac{1}{|\mathcal{I}(C)|} \int_{\mathcal{I}(C)} f\,dx \quad \text{and} \quad u_2(C) = \frac{1}{|\mathcal{O}(C)|} \int_{\mathcal{O}(C)} f\,dx$$

are the *averages* of the image inside and outside of C respectively.
Minimizing this functional with respect to C yields a segmentation of the image f.

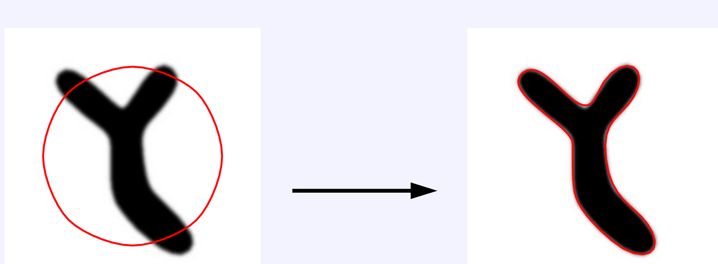


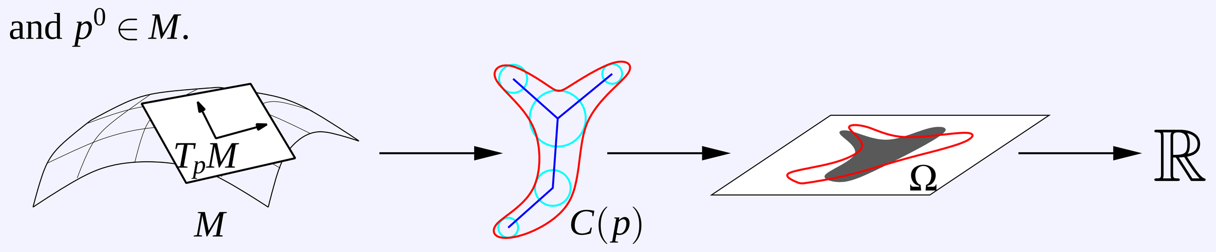
Image segmentation with the simplified Mumford-Shah functional.

Minimization over the Shape Manifold

We define the functional R_{α,p^0} as

$$\begin{aligned} R_{\alpha,p^0} : \quad M \quad &\to \mathcal{C}^1([0,1],\Omega) \to \mathbb{R} \,, \\ \underbrace{p}_{\text{shape}} &\mapsto \underbrace{C(p)}_{\text{shape boundary}} \mapsto \underbrace{I\big(C(p)\big)}_{\text{M-S functional}} + \underbrace{\frac{\alpha}{2} d_M(p^0, p)^2}_{\text{regularization}} , \end{aligned}$$

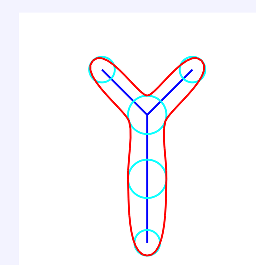
for $\alpha \geq 0$ and $p^0 \in M$.



The functional R_{α,p^0} maps M to $\mathbb{R}$.

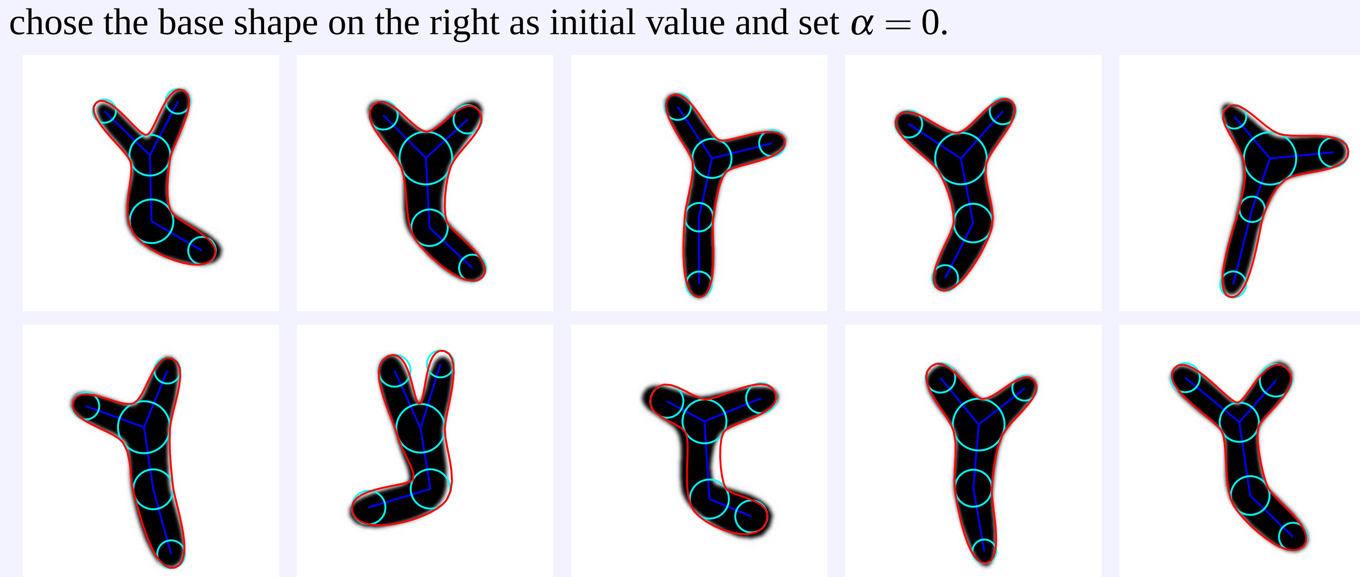
If f is continuous, then $\mathbb{R}_{\alpha,p^0}$ is differentiable. In this case, for every $p \in M$, the derivative of R_{α,p^0} in p maps

$$\begin{aligned} \nabla_p R_{\alpha,p^0} : T_pM &\to \mathbb{R} \,, \\ v &\mapsto \langle \nabla_p R_{\alpha,p^0}, v \rangle_{T_pM} \,. \end{aligned}$$



Initial value.

Furthermore, if f is differentiable, then $\mathbb{R}_{\alpha,p^0}$ is twice differentiable.
We minimize R_{α,p^0} using a *nonlinear conjugated gradient method*. Below are the results for 10 different training images. To segment the training data we chose the base shape on the right as initial value and set $\alpha = 0$.



Segmentation $p_1, \ldots, p_{10}$ of 10 training images.

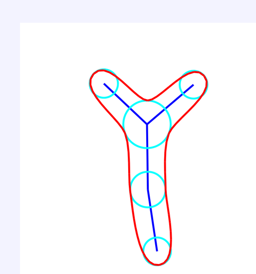
Principal Geodesic Analysis of the Training Data

Next we perform a Principal Geodesic Analysis of the models obtained from the training data [3, 4].

The Mean Shape

We define the *mean shape* μ as an element of the shape manifold M such that the squared distance to the data $p_1, \ldots, p_N$ is minimal:

$$\mu = \operatorname*{argmin}_{p \in M} \sum_{i=1}^{N} d_M(p_i, p)^2 \approx \operatorname{Exp}_{p^0}\left(\frac{1}{N}\sum_{i=1}^{N} \operatorname{Log}_{p^0}(p_i)\right).$$



The mean shape μ.

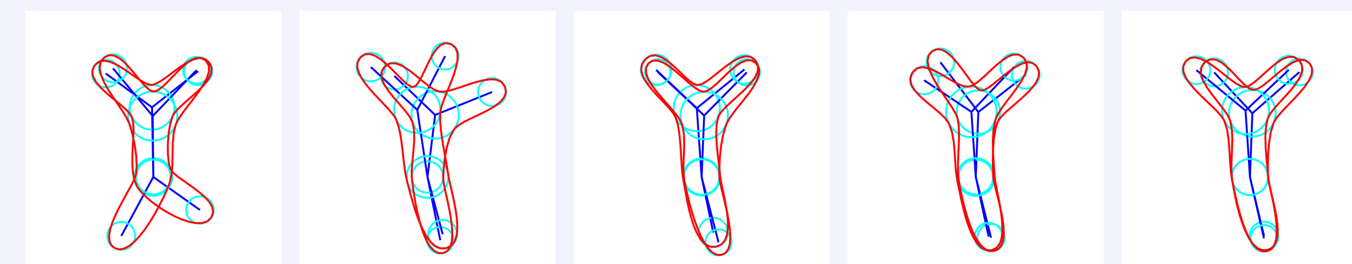
Principal Geodesics

The *principal geodesics* of the data $p_1, \ldots, p_N$ are 1-dimensional geodesic submanifolds of M such that the variances of the data within these submanifolds are maximal. We compute tangent vectors $v_1, \ldots, v_d \in T_\mu M$ (d is the dimension of M), which define the principal geodesics, by the following recursion:

$$v_1 = \operatorname*{argmax}_{\substack{v \in T_\mu M \\ |v|=1}} \sum_{i=1}^{N} d_M\big(\mu, \pi_{H_1(v)}(p_i)\big)^2 \approx \operatorname*{argmax}_{\substack{v \in T_\mu M \\ |v|=1}} \sum_{i=1}^{N} \langle v, \operatorname{Log}_\mu(p_i)\rangle^2 ,$$

$$\begin{aligned} v_k &= \operatorname*{argmax}_{\substack{v \in T_\mu M \\ |v|=1}} \sum_{i=1}^{N} d_M\big(\mu, \pi_{H_k(v)}(p_i)\big)^2 \\ &\approx \operatorname*{argmax}_{\substack{v \in T_\mu M \\ |v|=1}} \sum_{i=1}^{N} \left(\sum_{j=1}^{k-1} \langle v_j, \operatorname{Log}_\mu(p_i)\rangle^2 + \langle v, \operatorname{Log}_\mu(p_i)\rangle^2 \right). \end{aligned}$$

Here $\pi_{H_k(v)}$ denotes the projection onto the submanifold spanned by $v_1, \ldots, v_{k-1}$ and v.



Modes 1 – 5 of the training data.

Segmentation with Shape Prior

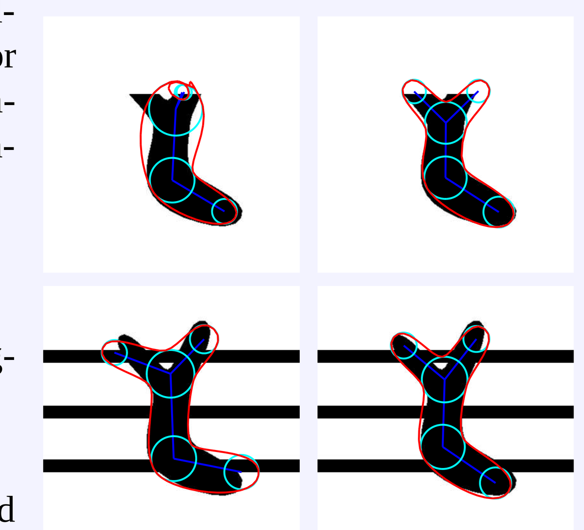
To segment incomplete image data using the a-priori knowledge (cf. [2]) of the mean shape μ we minimize $R_{\alpha,\mu}$ for $\alpha > 0$. On the right are examples for two different images. The segmentations in the left column were done without shape prior, i.e.

$$R_{0,-} \to \min ,$$

the segmentations in the right column with mean shape regularization

$$R_{\alpha,\mu} \to \min .$$

In both cases we chose μ as initial value for the conjugated gradient method.



Conclusion and Future Work

We segmented training images by minimizing the simplified Mumford-Shah functional over a shape manifold. Further, we performed a PGA on the result and used the mean shape as a prior to segment incomplete image data. In the future we want to work on the following subjects:

- Use not only the mean shape but also the knowledge of the significant principal geodesics to reconstruct perturbed image data.
- The metric on M is not unique. Try to improve the segmentation process by choosing more sophisticated distance measures.
- Apply the same ideas to the 3-dimensional case.

References and Acknowledgement

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