

# Why Should I Care About Lie Groups?

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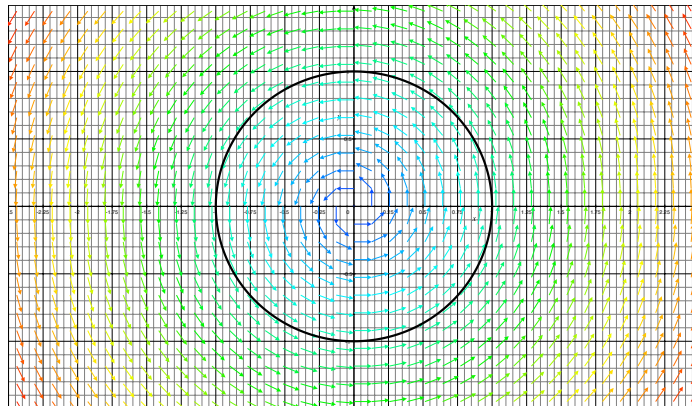
## Motivating Question

Consider the ordinary differential equation

$$\frac{d^2 x}{dt^2} = -\omega^2 x \quad \Longrightarrow \quad x(t) = r \cos(\omega t + \theta_0).$$

We may view this equation in *phase space*. Define  $y = -\frac{1}{\omega} \frac{dx}{dt}$ :

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



Could we have guessed these solutions using a rotational symmetry?



Marius Sophus Lie (1842 – 1899)

[www-groups.dcs.st-and.ac.uk/~history/Biographies/Lie.html](http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Lie.html)

# Review of Differential Geometry

## Differentiable Manifolds

An  $n$ -dimensional manifold is a triple  $(M, \{U_\alpha\}, \{\mu_\alpha\})$  satisfying:

- *Coordinate Charts and Maps:* For some countable indexing set  $I$ ,

$$M = \bigcup_{\alpha \in I} U_\alpha \quad \text{and} \quad \mu_\alpha : U_\alpha \hookrightarrow \mathbb{R}^n.$$

- *Locally Euclidean:* Each  $\mu_\alpha(U_\alpha)$  is a connected, open subset of  $\mathbb{R}^n$ ; and  $\mu_\beta \circ \mu_\alpha^{-1}$  is a smooth function.

$$\begin{array}{ccccc} \mathbb{R}^n & & M & & \mathbb{R}^n \\ \cup & \xrightarrow{\mu_\alpha^{-1}} & \cup & \xrightarrow{\mu_\beta} & \cup \\ \mu_\alpha(U_\alpha \cap U_\beta) & & U_\alpha \cap U_\beta & & \mu_\beta(U_\alpha \cap U_\beta) \end{array}$$

- *Hausdorff:* For distinct  $x \in U_\alpha$  and  $y \in U_\beta$  there exist open subsets

$$\begin{array}{l} \mu_\alpha(x) \in \mathcal{U}_\alpha \subseteq \mu_\alpha(U_\alpha) \\ \mu_\beta(y) \in \mathcal{U}_\beta \subseteq \mu_\beta(U_\beta) \end{array} \quad \text{such that} \quad \mu_\alpha^{-1}(\mathcal{U}_\alpha) \cap \mu_\beta^{-1}(\mathcal{U}_\beta) = \emptyset.$$

## Example: Circle

$T^1(\mathbb{R}) = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2 \right\}$  is a 1-dimensional manifold:

- *Coordinate Charts:* Write  $T^1(\mathbb{R}) = U_1 \cup U_2$  in terms of

$$U_1 = T^1(\mathbb{R}) - \{(0, -r)\} \quad \text{and} \quad U_2 = T^1(\mathbb{R}) - \{(0, r)\}.$$

- *Coordinate Maps:* By drawing lines  $y = tx + r$  and  $y = tx - r$ ,

$$\mu_1 : \quad U_1 \rightarrow \mathbb{R}^1, \quad (x, y) \mapsto t = \frac{y - r}{x} = -\frac{x}{y + r};$$

$$\mu_2 : \quad U_2 \rightarrow \mathbb{R}^1, \quad (x, y) \mapsto t = \frac{y + r}{x} = -\frac{x}{y - r}.$$

- *Smoothness:* The composition  $\mu_1 \circ \mu_2^{-1}$  sends

$$\mathbb{R}^1 \xrightarrow{\mu_2^{-1}} T^1(\mathbb{R}) \xrightarrow{\mu_1} \mathbb{R}^1, \quad t \mapsto \left( \frac{2t}{t^2 + 1} r, \frac{t^2 - 1}{t^2 + 1} r \right) \mapsto -\frac{1}{t}.$$

## Vector Fields on Manifolds

Let  $(M, \{U_\alpha\}, \{\mu_\alpha\})$  be an  $n$ -dimensional manifold. Let  $C$  be a *curve* on  $M$  i.e.,  $(C, \{V_\beta\}, \{\nu_\beta\})$  is a 1-dimensional manifold with  $C \subseteq M$ .

$$\begin{array}{ccccc}
 \mathbb{R}^1 & & C & & \mathbb{R}^n \\
 \cup & \xrightarrow{\nu_\beta^{-1}} & \cup & \xrightarrow{\mu_\alpha} & \cup \\
 \nu_\beta(U_\alpha \cap V_\beta) & & U_\alpha \cap V_\beta & & \mu_\alpha(U_\alpha \cap V_\beta)
 \end{array}$$

$\mu_\alpha \circ \nu_\beta^{-1}$  sends  $t$  to a vector  $\vec{x}(t) = x_1(t) \vec{e}_1 + x_2(t) \vec{e}_2 + \cdots + x_n(t) \vec{e}_n$ .  
 A *vector field* is a map  $\vec{v} : M \rightarrow \mathbb{R}^n$  given by *tangent vector* to  $C$  at  $P$ :

$$\vec{v}|_P = \frac{dx_1}{dt} \frac{\partial}{\partial x_1} + \frac{dx_2}{dt} \frac{\partial}{\partial x_2} + \cdots + \frac{dx_n}{dt} \frac{\partial}{\partial x_n} \quad \text{at} \quad t = \nu_\beta(P).$$

The *tangent bundle* of  $M$  is the set of all vectors  $\vec{v}|_P$  over all curves  $C$ :

$$T(M) = \bigcup_{P \in M} T_P(M) \quad \text{where} \quad T_P(M) = \left\{ \vec{v}|_P \mid P \in C \subseteq M \right\}.$$

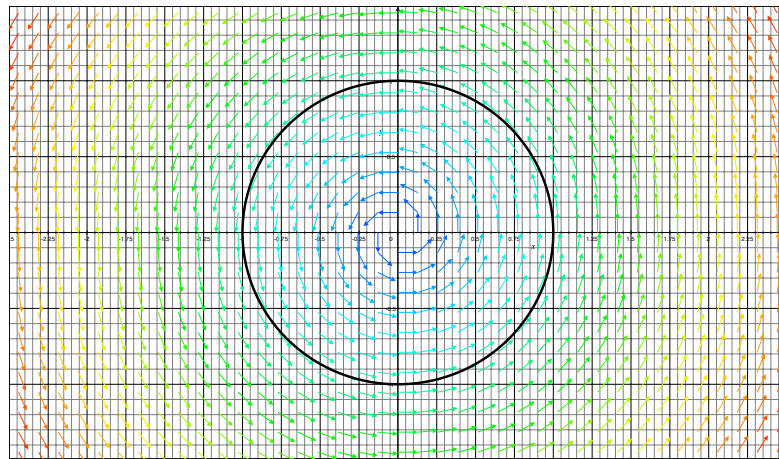
## Example: Circle

$$T^1(\mathbb{R}) = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2 \right\}, \quad r > 0;$$

is a 1-dimensional manifold. A curve  $C \subseteq T^1(\mathbb{R})$  has a parametrization  $\vec{x} : \mathbb{R}^1 \rightarrow \mathbb{R}^2$  which sends  $t \mapsto x(t) \vec{e}_1 + y(t) \vec{e}_2$  in terms of

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \cos \theta \\ r \sin \theta \end{bmatrix} \implies \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r \sin \theta \dot{\theta} \\ r \cos \theta \dot{\theta} \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \dot{\theta}$$

Every vector field  $\vec{v} = \dot{\theta} \vec{v}_\theta$  on  $T^1(\mathbb{R})$  is multiple of  $\vec{v}_\theta = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ .



Differential Equations  
on  
Manifolds

## Vector Flows on Manifolds

Let  $M$  be an  $n$ -dimensional manifold, and  $\vec{v} : M \rightarrow \mathbb{R}^n$  be a vector field. Define an *infinitesimal flow* as follows: Choose a point  $P \in M$ ; then  $\vec{v}|_P$  is the tangent vector to a curve  $C$  at  $P$ .

$$\vec{x} : \begin{array}{ccccc} \mathbb{R}^1 & & C & & \mathbb{R}^n \\ \cup & \xrightarrow{\nu^{-1}} & \cup & \xrightarrow{\mu} & \cup \\ I & & U & & \mathcal{U} \end{array} \quad t \mapsto P \mapsto \vec{x}(t)$$

Choose  $t + \epsilon \in I$  in an interval containing  $t = \nu(P)$ .

$$\exp : \mathbb{R}^1 \times M \rightarrow M, \quad (\epsilon, \vec{x}(t)) \mapsto \exp(\epsilon \vec{v}) \vec{x}(t) := \vec{x}(t + \epsilon).$$

- $\exp(0 \vec{v}) = 1$
- $\exp((\epsilon_1 + \epsilon_2) \vec{v}) = \exp(\epsilon_1 \vec{v}) \circ \exp(\epsilon_2 \vec{v})$
- $\left. \frac{d}{d\epsilon} \exp(\epsilon \vec{v}) \right|_0 = \vec{v}$  i.e.,  $\exp(\epsilon \vec{v}) = 1 + \epsilon \vec{v} + \dots$  as a formal sum.

## Example: Linear Vector Field

Let  $M = \mathbb{R}^n$ , and consider the vector field of “constant curvature:”

$$\vec{v} = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} x_j \right) \frac{\partial}{\partial x_i}.$$

In order to find the curve  $C \subseteq \mathbb{R}^n$ , consider the homogeneous equations

$$\frac{dx_i}{dt} = \sum_{j=1}^n a_{ij} x_j; \quad i = 1, 2, \dots, n.$$

The solutions can be found by exponentiating the matrix  $\mathbf{A} = [a_{ij}]$ :

$$\vec{x}(t) = \exp(t \mathbf{A}) \vec{b} = \left( \mathbf{1}_n + t \mathbf{A} + \dots + \frac{t^k}{k!} \mathbf{A}^k + \dots \right) \vec{b}.$$

The infinitesimal flow  $\exp : \mathbb{R}^1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an exponential:

$$(\epsilon, \vec{x}) \mapsto \exp(\epsilon \vec{v}) \vec{x} := \exp(\epsilon \mathbf{A}) \vec{x}.$$

## Continuous Symmetries of Manifolds

Let  $M$  be an  $n$ -dimensional manifold, and choose a collection  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$  of vector fields. The infinitesimal flows rotate  $M$ :

$$G = \left\{ \exp(\epsilon \vec{v}) : M \rightarrow M \mid \vec{v} \in \mathfrak{g} \right\} \quad \text{where} \quad \mathfrak{g} = \left\{ \sum_{i=1}^r \epsilon_i \vec{v}_i \mid \epsilon_i \in \mathbb{R}^1 \right\}.$$

From the formal identity  $g = \exp(\epsilon \vec{v}) = \mathbf{1} + \epsilon \vec{v} + \dots$  we see that

$$\begin{aligned} g \circ h \circ g^{-1} \circ h^{-1} &= \exp(\epsilon \vec{v}) \circ \exp(\epsilon \vec{w}) \circ \exp(-\epsilon \vec{v}) \circ \exp(-\epsilon \vec{w}) \\ &= \mathbf{1} + \epsilon^2 (\vec{v} \circ \vec{w} - \vec{w} \circ \vec{v}) + \dots \end{aligned}$$

We call the set  $G$  a (local) *Lie group* of  $M$ , the vector space  $\mathfrak{g}$  a *Lie algebra*, and the vector field  $[\vec{v}, \vec{w}] = \vec{v} \circ \vec{w} - \vec{w} \circ \vec{v}$  the *Lie bracket*.

- A (local) Lie group  $G$  is the collection of (infinitesimal) rotations.
- A Lie algebra  $\mathfrak{g}$  is a collection of differential equations.

# Review of Group Theory

# Groups

A *group* is a pair  $(G, \circ)$  satisfying:

- *Closure*: For each  $g, h \in G$ , there exists a unique element  $g \circ h \in G$ .
- *Associativity*:  $(g \circ h) \circ k = g \circ (h \circ k)$  for all  $g, h, k \in G$ .
- *Identity*: There exists a unique element  $e \in G$  such that  $g \circ e = e \circ g = g$  for all  $g \in G$ .
- *Inverses*: For each  $g \in G$ , there exists a unique element  $g^{-1} \in G$  such that  $g \circ g^{-1} = g^{-1} \circ g = e$ .

For any group  $(G, \circ)$ , we have two well-defined maps of interest:

$$\underbrace{\begin{array}{l} \circ : \\ G \times G \rightarrow G, \\ (g, h) \mapsto g \circ h; \end{array}}_{\text{Multiplication}}$$

and

$$\underbrace{\begin{array}{l} \iota : \\ G \rightarrow G, \\ g \mapsto g^{-1}. \end{array}}_{\text{Inversion}}$$

## Example: Matrix Groups

The *real general linear group* of degree 2 is defined as

$$GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{R}) \mid a_{11} a_{22} - a_{12} a_{21} \neq 0 \right\}.$$

- *Multiplication:* The map  $\circ : GL_2(\mathbb{R}) \times GL_2(\mathbb{R}) \rightarrow GL_2(\mathbb{R})$  sends

$$\left( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) \mapsto \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{bmatrix}.$$

- *Inversion:* The map  $\iota : GL_2(\mathbb{R}) \rightarrow GL_2(\mathbb{R})$  sends

$$g = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto g^{-1} = \frac{1}{a_{11} a_{22} - a_{12} a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

# Lie Groups

We say that  $G$  is an  $r$ -parameter Lie group if

- $(G, \{U_\alpha\}, \{\mu_\alpha\})$  is a  $r$ -dimensional manifold,
- $(G, \circ)$  is a group, and
- both multiplication  $\circ : G \times G \rightarrow G$  and inversion  $\iota : G \rightarrow G$  are smooth maps between manifolds.

**Remarks:** Let  $(G, \{U_\alpha\}, \{\mu_\alpha\})$  and  $(H, \{V_\beta\}, \{\nu_\beta\})$  be  $r$ - and  $s$ -manifolds, respectively.

- The Cartesian product  $G \times H$  is an  $(r + s)$ -manifold.
- A map  $\varphi : G \rightarrow H$  is smooth if  $\phi = \nu_\beta \circ \varphi \circ \mu_\alpha^{-1}$  is smooth:

$$\phi : \quad \mathbb{R}^r \xrightarrow{\mu_\alpha^{-1}} G \xrightarrow{\varphi} H \xrightarrow{\nu_\beta} \mathbb{R}^s$$

## Example: Matrix Groups

The real general linear group  $G = GL_2(\mathbb{R})$  is a 4-parameter Lie group:

- $(G, \circ)$  is a group.
- Each invertible  $2 \times 2$  matrix has an *Iwasawa Decomposition*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \underbrace{\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}}_{\text{affine transformation}} \underbrace{\begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix}}_{\text{rotation}}$$

in terms of the four parameters

$$m = \frac{a_{11} a_{21} + a_{12} a_{22}}{a_{12}^2 + a_{22}^2} \qquad r = \sqrt{a_{12}^2 + a_{22}^2}$$

$$b = \frac{a_{11} a_{22} - a_{12} a_{21}}{a_{12}^2 + a_{22}^2} \qquad \theta = \arctan \frac{a_{21}}{a_{22}}$$

Hence  $G$  is a 4-dimensional manifold.

- Multiplication  $\circ : G \times G \rightarrow G$  and inversion  $\iota : G \rightarrow G$  are smooth.

# Exercise

## What is the Lie Algebra of $M = \mathbb{R}^1$ ?

$M = \mathbb{R}^1$  is a 1-dimensional manifold. Every vector field is in the form  $\vec{v} = \dot{x} \frac{\partial}{\partial x}$  for some function  $x = x(t)$ . What is the Lie algebra  $\mathfrak{g}$  of  $M$ ?

**Theorem (Lie).** *The Lie algebra of  $M = \mathbb{R}^1$  is a 3-dimensional real vector space. Explicitly,*

$$\begin{aligned} \mathfrak{g} &= \left\{ \epsilon_1 \frac{\partial}{\partial x} + \epsilon_2 x \frac{\partial}{\partial x} - \epsilon_3 (1 + x^2) \frac{\partial}{\partial x} \mid \epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}^1 \right\} \\ &\cong \left\{ \epsilon_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \epsilon_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \epsilon_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mid \epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}^1 \right\}. \end{aligned}$$

**Remark:**  $\mathfrak{g}$  is equivalent to the set of  $2 \times 2$  matrices with trace zero.

## Step #1: Iwasawa Decomposition

*Proof:* The automorphisms of  $M = \mathbb{R}^1$  are *Möbius* or *fractional linear transformations*:

$$x \mapsto \frac{a_{11}x + a_{12}}{a_{21}x + a_{22}} \quad \text{where} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in GL_2(\mathbb{R}).$$

Using the Iwasawa Decomposition

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix}$$

we can determine the Lie group  $G$  of  $M$ . The scalar  $r$  acts trivially, so the group is a 3-dimensional manifold. The (connected component of the) Lie group is  $G^\circ \cong SL_2(\mathbb{R})$ .

## Step #2: Affine Transformations

Denote  $\epsilon_1 = (\log m / (m - 1)) b$  and  $\epsilon_2 = \log m$ .

- The equation  $\frac{dx}{dt} = \epsilon_1 + \epsilon_2 x$  has solution  $x(t) = m^t x_0 + \frac{m^t - 1}{m - 1} b$ .

$$\exp \left( \epsilon_1 \frac{\partial}{\partial x} + \epsilon_2 x \frac{\partial}{\partial x} \right) x(t) = x(t + 1) = m \cdot x(t) + b.$$

- We have the matrix exponential

$$\exp \left( \epsilon_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \epsilon_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} \epsilon_2 & \epsilon_1 \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}.$$

This shows that the infinitesimal flow is the matrix exponential

$$\exp \left( \epsilon_1 \frac{\partial}{\partial x} + \epsilon_2 x \frac{\partial}{\partial x} \right) = \exp \left( \epsilon_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \epsilon_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}.$$

## Step #3: Rotations

Denote  $\epsilon_3 = \theta$ .

- $\frac{dx}{dt} = -\epsilon_3 (1 + x^2)$  has solution  $x(t) = \frac{\cos \theta t \cdot x_0 - \sin \theta t}{\sin \theta t \cdot x_0 + \cos \theta t}$ .

$$\exp \left( -\epsilon_3 (1 + x^2) \frac{\partial}{\partial x} \right) x(t) = x(t + 1) = \frac{\cos \theta \cdot x(t) - \sin \theta}{\sin \theta \cdot x(t) + \cos \theta}.$$

- We have the matrix exponential

$$\exp \left( \epsilon_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 0 & -\epsilon_3 \\ \epsilon_3 & 0 \end{bmatrix}^k = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

This shows that the infinitesimal flow is the matrix exponential

$$\exp \left( -\epsilon_3 (1 + x^2) \frac{\partial}{\partial x} \right) = \exp \left( \epsilon_3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Questions?