Applying Numerical Algebraic Geometry to Partial Differential Equations

Wenyuan Wu and Greg Reid

email: wwu25@uwo.ca reid@uwo.ca

0. Background

- PDE occurs frequently in applications.
- We algorithmically study such systems, eg. to obtain global info about their solution spaces and construct formal power series solutions.
- Regard PDE theory from the point of view of geometry, so-called Jet Geometry of PDE, initiated by Sophus Lie and developed by Elie Cartan and others.

Apply a finite number of differentiations and eliminations (Diff. Elim. Methods)

Goal: to yield all missing constraints; the Zariski Closure of the union of all integral manifolds = algebraic varieties (local solvability).

Advantages: eases the application symbolic and numeric solution techniques; enables the statement of EU Theorems for their solutions.

1. Motivation of Our Work

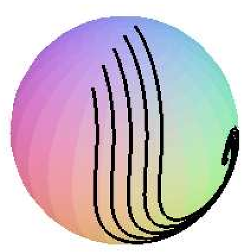
- Despite considerable progress in the exact case by Boulier, Hubert, Mansfield, Seiler etc. these methods are very expensive !
- This is a continuation of symbolic-numeric methods for PDE begun in Wittkopf & Reid [1] , Reid, Smith & Verschelde [2], Reid, Verschelde, Wittkopf & Wu [4].
- Numerical Algebraic Geometry[3] provides us a numerical way to study varieties. We replace these exact algebraic methods with a numerical method.
- Exploit linearity which always appears after prolongation of PDE.

2. Formal Jet Theory

Consider q-th order PDE system $R = (R^1, ..., R^k) = 0$ with indep vars $x = (x_1, x_2, ..., x_n)$ and dep vars $u(x) = (u^1(x), u^2(x), ..., u^m(x))$. Denoting $\underset{r}{u}$ as the formal (jet) variables corresponding to r-th order partial derivatives of $u(x)$ the jet variety is:

$$V(R) := \{(x, u, \underset{1}{u}, ..., \underset{q}{u}) \in J^q : R(x, u, \underset{1}{u}, ..., \underset{q}{u}) = 0\},$$
$$\text{where } J^q = \mathbb{C}^{N_q}, N_q = n + m\binom{n+q}{q}.$$

Example (Tuoemela & Arponen): $V(R) = \{(x, u, u_x) : u_x^2 + u^2 + x^2 = 1\}$



All the extended solutions lying in $V(R)$ and for any regular point on $V(R)$ there exists an extended solution passing it. This property is called local solvablility.

References

[1] Wittkopf and Reid. **Fast differential elimination in C: The CDiffElim environment.** *Computer Physics Communications*, 139: 192–217, 2001.

[2] Reid, Smith, and Verschelde. **Geometric completion of differential systems using numeric-symbolic continuation.** *SIGSAM Bulletin* 36(2):1–17, 2002.

[3] Sommese and Wampler. **The Numerical solution of systems of polynomials arising in engineering and science.** World Scientific Press, Singapore, 2005.

3. Numerical Completion methods for PDE

R is a PDE system with order q, the prolongation of R: $\mathbf{D}R = \{\mathcal{S} \cdot \underset{q+1}{u} + \mathbf{r}, R\}$

Cartan-Kuranishi's Involution test requires 2 conditions:

- $\overline{\boldsymbol{\pi}\mathbf{D}R} = V(R)$ (termination since polynomial ring is Noetherian.)
- Symbol is involutive (Spencer Cohomology groups will vanish after finite prolongations [δ-Poincaré lemma]).

Projection is done by diff. elim. with ranking: $\underset{0}{u} \prec \underset{1}{u} \prec ... \prec \underset{q}{u} \prec ...$

If $\boldsymbol{\pi}\mathbf{D}R \neq V(R)$, then let $R = \{R$, new constraints $\}$ and repeat until conditions are satisfied. It will terminate after finite steps.

4. Using Witness Points

To check the "new constraints":

- Symbolic methods: (radical) ideal membership test by GB or Δ-decomp (can be expensive).
- new method: rank test at witness points to detect the existence of "new eqn".

Theorem 1 *For any* $p \in W(R)$, $\text{rank}(\mathcal{S}_p) = \text{rank}([\mathcal{S}_p, \boldsymbol{r}_p]) \Leftrightarrow V(R) = \overline{\boldsymbol{\pi}\mathbf{D}R}$.

Here $\overline{A}$ means the Zariski Closure of the set A. If some witness points satisfy the rank condition but others do not, then split will happen and numerical irreducible decomposition is needed.

5. Numerical Differential Elimination Method [5]

Looking for column vector f in order q, such that $f^t \cdot \mathcal{S} = 0$, then

$$f^t \cdot (\mathcal{S} \cdot \underset{q+1}{u} + \mathbf{r}) = f^t \cdot \mathbf{r} = 0$$

Theorem 2 *Let* $F := \mathit{Null}(\mathcal{S}^t)$ *and* $P := \boldsymbol{r}^t \cdot F$, *then:*

1. $\boldsymbol{\pi}\mathbf{D}R \subseteq V(R) \cap V(P)$

2. $\forall p \in W(V(R) \cap V(P))$, $\text{rank}(\mathcal{S}_p) = \text{rank}([\mathcal{S}_p, \boldsymbol{r}_p]) \Rightarrow \overline{\boldsymbol{\pi}\boldsymbol{D}R} = V(R) \cap V(P)$

$W(V(R) \cap V(P))$ can be computed by diagonal homotopy methods.

6. Computing f by Polynomial Matrix [6]

Let $\mathcal{R} = K[x_1, ..., x_s]$, K is a field and $Q(\mathcal{R})$ is the quotient field of $\mathcal{R}$.

Polynomial Matrix: The set of all $m \times n$ matrices with entries in $\mathcal{R}$ will be denoted by $M^{m \times n}(\mathcal{R})$.

Rank: The (column) rank of polynomial matrix $A \in M^{m \times n}(\mathcal{R})$ is the maximum number of linearly independent column vectors of A. At a generic point $x_0 \in \mathbb{C}^s$, $\text{rank}(A) = \text{rank}(A_{x_0})$.

Null-space: there exist $r = n - \text{rank}(A)$ linearly independent polynomial vectors $\{f_i\}$ such that $Af_i = 0^{m \times 1}$. $f_1, ..., f_r$ generates a linear space over $Q(\mathcal{R})$.

Main Idea: There is a natural bijection: $M^{m \times n}(K[x]) \leftrightarrow M^{m \times n}(K)[x]$. By the bijection $A \rightarrow A(x)$ and $f \rightarrow f(x)$, write them in polynomial form: $A(x) = \sum_{i=1}^{T(d_1)} A_i x^{\alpha_i}$ and $f(x) = \sum_{j=1}^{T(d_2)} f_j x^{\beta_j}$. So $Af = 0 \Leftrightarrow A(x)f(x) = 0 = \sum_{k=1}^{T(d_1+d_2)} C_k x^{\gamma_k}$. Each C_k which depends on the coefficients $\{A_i\}, \{f_j\}$ must be 0. Naturally, we write the coefficients of $f(x)$ as a vector: $v_f := [f_1, ..., f_{T(d_2)}]^t$. Then $\exists\, M_A$, s.t. $M_A v_f = (C_1, ..., C_k, ...)^t$. M_A (Sylvester matrix of $\mathcal{S}$) only depends on $\{A_i\}$, so we can solve this homogenous linear eqn to construct f.

7. Simple Example

We implement the methods in Maple 10 using Jan Verschelde's PHCpack.

The input system is $\langle u_t, v_t - u(u-1), u(v-1)\rangle$. First add $\partial_t(u(v-1)) = (v-1)u_t + uv_t$ into the system:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ v-1 & u \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ u(u-1) \\ 0 \end{pmatrix} \quad (1)$$

with the constraint: $u(v-1) = 0$. the witness points: $(0, v^0)$ and $(u^0, 1)$. At $(u^0, 1)$, ranks are not equal. Compute Null Space basis of symbol which is $(1-v, -u, 1)$ and new eqn is $(1-v, -u, 1) \cdot (0, u(u-1), 0)^t = -u^2(u-1)$. Also add the prolongation of new one to the system. Then rank test shows no new eqns and full rank symbol. Hence the method terminates.

8. Random square PDE

Theorem 3 *Consider a system of* m *random PDE* $R^1, R^2, ..., R^m$ *in* $\mathbb{C}[u, \underset{1}{u}, ..., \underset{q}{u}]$. *in* m *dependent variables* u^1, u^2, ... , u^m *and* n *independent variables* $x^1, ..., x^n$ *where each PDE has order* q. *Then with probability* 1 *the system is involutive.*

Experimental test for 2 eqns. and 2 dependent variables.

1. $R = \{R^1(u_x, u_y, v_x, v_y, u, v), R^2(u_x, u_y, v_x, v_y, u, v)\}$ randomly with degree 2.
2. $DR = \{D_x R^1, D_y R^1, D_x R^2, D_y R^2, R^1, R^2\}$.
3. R' = random linear combination of DR, 6 eqns with order 2.

Note that R is involutive by Theorem 3. Now we will show R' is involutive. Rank test shows there are new constraints. Construct the projected eqns $new = \{S^1, S^2\}$ by NS-basis. $DR' = \{R', new, D(new)\}$. The rank test shows no new eqns and we can check that the Symbol is involutive by Cartan's test. So R' is involutive.

Summary of experimental results:

- Previous methods **explode** on this problem.
- New method can handle such systems even when the degree is 5.

9. Conclusion

- Successfully exploit leading linear structure of PDEs using polynomial matrix.
- Combined with numerically stable homotopy techniques, we get a new differential elimination method for approx. systems.
- Future work includes exploiting the sparse structure of M_A (Sylvester matrix of $\mathcal{S}$), e.g. block toeplitz structure and Combined with some optimization techniques, apply new method to large sparse systems from realistic problems.

More References

[4] Reid, Verschelde, Wittkopf and Wu. **Symbolic-Numeric Completion of Differential Systems by Homotopy Continuation.** *Proc. ISSAC 05, pp269-276*, 2005.

[5] Wenyuan Wu and Greg Reid. **Application of Numerical Algebraic Geometry and Numerical Linear Algebra to PDE.** *Proc. ISSAC 06 pp345-352*, 2006.

[6] Wenyuan Wu. **Computing the Rank and Null-space of Polynomial Matrices.** *Preprint*.

University of Western Ontario, Department of Applied Mathematics, London, ON. N6G 5B7, Canada