The concept of graphs is extraordinarily simple, which explains their wide applicability. It dates back to 1736, when Euler formalized the now-famous "bridges of Königsberg" problem. Graph theory has now grown to touch almost every mathematical discipline, and it likewise borrows from elsewhere tools for its own problems. Anyone who delves into graph theory will see that the lifeblood of graph theory is the abundance of tricky questions and clever and course, general results that systematize the subject, but we also find an emphasis on the problems over building machinery for its own sake.

Graph theory representation of Königsberg bridges problem.

For quite long graph theory was regarded as a branch of topology concerned with 1-dimensional spaces; however this view has faded away. The only remainder of the topological past is the topological graph theory that primarily deals with drawing of graphs on surfaces. The most famous result of topological graph theory is the proof of the four-color conjecture (every political map on a sphere can be colored into four colors given that each country consists of only one piece).

Now, a (finite) graph is usually thought of as a subset of pairs of elements of a finite set, generally as a family of arbitrary sets in the case of hypergraphs. For instance, Ramsey graph theory deals with determining how disordered can graphs be. The central result is the Ramsey theorem which states that one can always find many vertices that are either all connected or not connected to each other, given that the graph is sufficiently large. The other result is Szemerédi regularity lemma.

The four-color conjecture mentioned above is one of the problems in graph coloring. We can color a graph, but the most common are vertex coloring and edge coloring. In these cases, we color vertices (edges) of a graph so that no two vertices of the same color are adjacent (two vertices of the same color share a common vertex). The most common problem is to find a coloring using as few colors as possible. Such problems often arise in scheduling problems.

Graph theory benefits greatly from interaction with other fields of mathematics. For example, methods have become the standard tool in the arsenal of graph theorists, and random graphs into a full-fledged branch of its own.

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1 Prerequisites

Random variables, expected value

2 Summary

Integration can be seen as a kind of limit operation— we approximate a given function by a sequence of step functions, etc. This section will treat the topic of interchanging integration with other limit operations. The centerpiece of this section is Lebesgue’s Dominated Convergence Theorem, which has been called the swiss army knife for integration problems. Fatou’s Lemma and the monotone convergence theorem are also quite useful, and they are proved in this section as well.

3 Integration and Limit

Define $X_n$ on $[0, 1]$ as $X_n = n1_{(0,1/n)}$. That is, $X_n$ is $n$ with probability $1/n$ and 0 otherwise. Then

$$\lim_{n \to \infty} E(X_n) = \lim_{n \to \infty} 1 = 1 \neq 0 = E(0) = E\left(\lim_{n \to \infty} X_n\right)$$

(1)

This example shows that integration and limit cannot always be exchanged. However, there are circumstances which allow one to interchange limits.

Theorem 1 (Monotone Convergence Theorem) If $0 \leq X_n \uparrow X$ then $E(X_n) \uparrow E(X)$.

Proof: Since $E(X_n) \leq E(X_{n+1})$, there is $\alpha \in [0, \infty]$ such that $E(X_n) \to \alpha$ as $n \to \infty$. Furthermore, since $X_n \leq X$ we have $E(X_n) \leq E(X)$, and thus $\alpha \leq E(X)$. Let $S$ be any simple random variable such that $0 \leq S \leq X$ and let $c$ be a constant $0 < c < 1$. 