# Representations of Positive Polynomials and SDP-relaxations exploiting sparsity

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# Outline

- Problem statement
- Specialized Positivstellensatz: part I: $(X, Y, Z)$
- Specialized Positivstellensatz: part II: general case
- Sparse SDP-relaxations for global polynomial optimization

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# Problem statement

Let $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_m)$ and $Z = (Z_1, \ldots, Z_p)$.

$$\begin{aligned} \mathbf{K} &:= \{(x, y, z) \in \mathbb{R}^{n+m+p} \,:\, (x, y) \in \mathbf{K}_{xy}\,; \quad (y, z) \in \mathbf{K}_{yz}\,\} \\ \mathbf{K}_{xy} &:= \{(x, y) \in \mathbb{R}^{n+m} \,:\, g_j(x, y) \geq 0, \quad j \in J_{xy}\,\} \\ \mathbf{K}_{yz} &:= \{(yz) \in \mathbb{R}^{m+p} \,:\, h_j(y, z) \geq 0, \quad j \in J_{yz}\,\} \end{aligned}$$

and let $f \in \mathbb{R}[X, Y] + \mathbb{R}[Y, Z]$.

• So in the definition of **K** and $f$ there is **NO COUPLING** of variables $X$ and $Z$.

• Can we have a **specialized representation** of $f > 0$ on **K** that preserves this coupling pattern?

• If yes, can we extend to **more than two** subsets of variables $(X, Y)$ and $(Y, Z)$?

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# A specialized representation : part I

Let $P(g) \subset \mathbb{R}[X,Y]$ and $Q(g) \subset \mathbb{R}[X,Y]$ be the preordering and quadratic module generated by $(g_j)_{j\in J_{xy}}$.

That is:

$$P(g) := \left\{ \sum_{J\subset J_{xy}} \sigma_J \prod_{j\in J} g_j \; : \quad \sigma_J \in \Sigma^2_{xy} \right\}$$

$$Q(g) := \left\{ \sigma_0 + \sum_{j\in J_{xy}} \sigma_j \, g_j \; : \quad \sigma_0, \, \sigma_j \in \Sigma^2_{xy} \right\}$$

Similar definitions for $P(h)$, $Q(h) \subset \mathbb{R}[Y,Z]$.

**Theorem : Las (2006)**

Let $f \in \mathbb{R}[X, Y] + \mathbb{R}[Y, Z]$, and let **K** be compact.

**(a)** If $f$ is positive on **K** then $f \in P(g) + P(h)$.

**(b)** If $N - \|(X, Y)\|^2 \in Q(g)$ and/or $N - \|(Y, Z)\|^2 \in Q(h)$ for some scalar $N$, and if $f$ is positive on **K**, THEN

in (a) one may replace $P(g)$ with $Q(g)$ and/or $P(h)$ with $Q(h)$.

**(a)** is a **specialized version** of Schmüdgen's Positivstellensatz with **no coupling** between $X$ and $Z$ in the representation of $f$.

**(b)** is a **specialized version** of Putinar, and Jacobi & Prestel's Positivstellensatz.

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Fact:

Let $\psi$ be a probability measure on $X \times Y$, a *cartesian product* of Borel spaces. Then one may disintegrate $\psi$ into

- a stochastic kernel $\psi_{X|Y}(dx\,|\,y)$ on $X$ given $Y$, and
- a probablity measure $\psi_Y(dy)$ on $Y$, and so

$$\psi(A \times B) \;=\; \int_B \psi_{X\times Y}(A\,|\,y)\,\psi_Y(dy), \qquad A \in \mathcal{B}(X),\; B \in \mathcal{B}(Y)$$

$\psi_Y$ is the marginal of $\psi$ on $Y$ whereas $\psi_{X\times Y}$ is the conditional probability.

$\psi_{X|Y}(dx\,|\,y)$ is a stochastic kernel on $X$ given $Y$ if
- for every fixed $y \in Y$, $\psi_{X|Y}(dx\,|\,y)$ is a p.m. on $X$
- for every fixed $A \in \mathcal{B}(X)$, the function $y \mapsto \psi_{X|Y}(A\,|\,y)$ is a bounded Borel measurable on $Y$.

Then the proof relies on the following crucial property:
Let $\nu\,(d(x,y))$ and $\varphi\,(d(y,z))$ be probability measures with same marginal on $y$, i.e.,

$$\nu_y\,(dy)\,\equiv\,\varphi_y\,(dy)\,=:\,\psi(dy),$$

then one may construct $\mu\,(d(x,y,z))$ such that

$$\mu_{xy}\,(d(x,y))\,\equiv\,\nu\,(d(x,y))\,;\quad \mu_{yz}\,(d(y,z))\,\equiv\,\varphi\,(d(y,z)).$$

In fact $\mu$ is defined via:

$$\mu\,(A\times B\times C)\,=\,\int_B \nu_{x|y}\,(A\,|\,y)\,\varphi_{z|y}(C\,|\,y)\,\psi\,(dy)$$

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Observe that apart from the usual compactness assumption on $\mathbf{K}$ and on its representation, no other assumption on the coupling pattern of variables is needed. That is:

If there is no coupling between $X$ and $Z$, then a specialized Positivstellensatz with no coupling between $X$ and $Z$ also exists!

.... Yields specialized SDP-relaxations for polynomial optimization with significant computational savings ... More later .....

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# part II: general case

Assume that $\mathbf{K} \subset \mathbb{R}^n$ is compact and one knows some scalar $M$ such that $M - \|X\|^2 > 0$ on $\mathbf{K}$.

Let $I = \{1, \ldots, n\} = \bigcup_{i=1}^{p} I_p$   Let $\mathbb{R}[X(I_k)] = \mathbb{R}[X_i \mid i \in I_k]$.

**Sparsity pattern:**

- $f \in \mathbb{R}[X(I_1)] + \ldots + \mathbb{R}[X(I_p)]$.
- $\forall j = 1, \ldots, m, \quad g_j \in \mathbb{R}[X(I_{k(j)})]$ for some $k(j) \in \{1, \ldots, p\}$.

In other words ... a monomial $X^\alpha$ appears in $f$ or $g_j$ if and only if $\mathrm{support}(\alpha) := \{i \mid \alpha_i \neq 0\} \subseteq I_k$ for some $k$.

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# A sparse Positivstellensatz

- Add the p redundant quadratic constraints

$$g_{m+k}(X) = M - \|X(I_k)\|^2 \geq 0, \qquad k = 1, \ldots, p$$

in the definition of the constraint set **K**. $\quad (g_{m+k} \in \mathbb{R}[X(I_k)].)$

$$\text{Let} \quad \mathcal{G}_k := \{\, g_j \; : \; g_j \in \mathbb{R}[X(I_k)] \,\} \quad k = 1, \ldots, p.$$

**Theorem : Las (2006)**

Let $f \in \sum_{k=1}^{p} \mathbb{R}[X(I_k)]$, let $\mathbf{K} \subset \mathbb{R}^n$ be compact, and assume that

$$\forall k = 2, \ldots, p, \quad I_k \bigcap \left( \bigcup_{l=1}^{k-1} I_l \right) \subseteq I_s, \qquad \text{for some } s \leq k-1$$

Then $\quad f \quad$ positive $\quad$ on $\quad$ **K** $\quad \Rightarrow \quad f \in \sum_{k=1}^{p} Q(\mathcal{G}_k)$.

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Then $f$ positive on $\mathbf{K}$ $\Rightarrow$ $f \in \sum_{k=1}^{p} Q(\mathcal{G}_k)$.

That is

$$f = \sum_{k=1}^{p} \left( \sigma_{0k} + \sum_{g_j \in \mathcal{G}_k} \sigma_{jk}\, g_j \right), \qquad \sigma_{jk} \in \mathbb{R}[X(I_k)]^2$$

The hypothesis

$$\forall k = 2, \ldots, p, \quad I_k \bigcap \left( \bigcup_{l=1}^{k-1} I_l \right) \subseteq I_s \qquad \text{for some } s \leq k-1$$

is the Running Intersection Property (RIP) in graph theory.

Used in Bayesian networks for efficient global update of information from local modifications at some nodes of the network...

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The proof relies on the following crucial property: Let $\nu_k(dX_i : i \in I_k)$, $k = 1, \ldots, p$ be probability measures with consistent marginals, i.e., whenever $I_k \cap I_j \neq \emptyset$ ..

$$\nu_{kj}(dX_i : i \in I_k \cap I_j) \equiv \nu_{jk}(dX_i : i \in I_k \cap I_j) =: \nu_{jk}$$

Then, under RIP, one may construct from the local $\nu_k$'s ... a **global** prob. measure $\mu(dX_i : i = 1, \ldots, n)$ on $\mathbb{R}^n$, such that

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# SPARSE SDP-relaxations for global optimization

For the global polynomial optimization problem

$$f^* = \min_x \{ f(x) \; : \quad g_j(x) \geq 0, \quad j = 1, \ldots, m \}$$

**REPLACE** the standard hierarchy of SDP-relaxations

$$\mathbf{Q}_r^* \begin{cases} \max\limits_{\lambda, q_0, \ldots, q_m} \quad \lambda \\ \\ f - \lambda = q_0 + \sum_{j=1}^{m} q_j \, g_j \\ \\ q_j \quad s.o.s. \in \mathbb{R}[X] \quad \deg(q_j \, g_j) \leq 2r, \quad \forall j = 0, \ldots, m \end{cases}$$

## WITH the hierarchy of SPARSE SDP-relaxations

$$
\mathbf{Q}_r^* \left\{ \begin{array}{l}
\displaystyle\max_{\lambda, q_0, \ldots, q_m} \quad \lambda \\[2em]
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q_j \quad s.o.s. \; \in \mathbb{R}[X(I_{k(j)})] \quad \deg(q_j\, g_j) \leq 2r, \quad \forall j = 1, \ldots, m \\[2em]
\displaystyle q_0 = \sum_{k=1}^{p} q_{0k}, \quad q_{0k} \quad s.o.s. \; \in \mathbb{R}[X(I_k)] \quad \deg(q_{0k}) \leq 2r
\end{array} \right.
$$

# Convergence

**Theorem :** If the sparsity pattern $I = \bigcup_{k=1}^{p} I_k$ satisfies the **running intersection property** .....

$$\forall k = 2, \ldots, p, \quad I_k \bigcap \left( \bigcup_{l=1}^{k-1} I_l \right) \subseteq I_s, \qquad \text{for some } s \leq k-1$$

.... then $\min \mathbf{Q}_r \uparrow f^*$ as $r \to \infty$.

# Sketch of the proof

Recall that one has

$$f^* = \min_x \{ f(x) \,:\, g_j(x) \geq 0, \quad j = 1, \ldots, m \} = \min_x \{ f(x) \,:\, x \in \mathbf{K} \}$$
$$= \min_\mu \{ \int f \, d\mu \,:\, \quad \mu \in \mathcal{P}(\mathbf{K}) \}$$

Let $\quad \mathcal{G}_j := \{ g_k \,:\, \quad g_k \in \mathbb{R}[X(I_j)] \} \quad j = 1, \ldots, p.$

$$\mathbf{K}_j := \{ x \in \mathbb{R}^{n_j} \,:\, \quad g_k(x) \geq 0, \quad g_k \in \mathcal{G}_j \}$$

$$\rho^* := \min_{\mu_1, \ldots, \mu_p} \left\{ \sum_{j=1}^{p} \int f_j \, d\mu_j \,:\, \quad \mu_j \in \mathcal{P}(\mathbf{K}_j) \right\} \leq f^*$$

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$$\begin{aligned} f^* &= \min_x \{ f(x) \, : \, g_j(x) \geq 0, \quad j = 1, \ldots, m \} = \min_x \{ f(x) \, : \, x \in \mathbf{K} \} \\ &= \min_\mu \{ \int f \, d\mu \, : \quad \mu \in \mathcal{P}(\mathbf{K}) \} \end{aligned}$$

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In computing $\rho^*$ ... one has replaced the single prob. measure $\mu$ on the BIG space $\mathbb{R}^n$ .... with

$p$ measures $\mu_1, \ldots, \mu_p$ on the SMALLER spaces $\mathbb{R}^{n_1}$, ..., $\mathbb{R}^{n_p}$

So in general $\rho^* \leq f^*$ ... BUT under the running intersection property RIP .....

from the $\mu_j$'s ... one may construct a global prob. measure $\mu$ on the whole $\mathbb{R}^n$, such that the marginal of $\mu$ on $\mathbf{K}_j$ is $\mu_j$, for all $j = 1, \ldots, p$. And therefore,

$$\rho^* = \sum_{j=1}^{p} \int f_j \, d\mu_j = \int f \, d\mu = f^*$$

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# STANDARD vs SPARSE SDP-relaxations

Let $\kappa := \max_{k=1,\ldots,p} \text{card } I_k$.

| | |
|---|---|
| STANDARD SDP-relaxation $\mathbf{Q}_r$ | SPARSE SDP-relaxation $\mathbf{Q}_r$ |
| $O(n^{2r})$ VARIABLES | $p\,O(\kappa^{2r})$ VARIABLES |
| $m+1$ LMIs of size $O(n^r)$ | $m+2p$ LMIs of size $O(\kappa^r)$ |

Hence, SPARSITY yields a **big saving** if $\kappa$ is **small**, as is frequently the case in applications with large scale problems.

Tests by Kojima's group with the specialized software SPARSEPOPT on a sample of nontrivial and nonconvex large scale problems, are very encouraging:

For instance, they could solve problems with up to $n = 500$ and even $n = 1000$ variables with $\kappa$ in the range $[3, 7]$.

For much smaller problems ... one cannot even implement the first standard SDP-relaxation!