

## GRÖBNER BASES AND INTEGER PROGRAMMING: PROBLEM SESSION II

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$$\text{Let } \mathbf{COIN} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 10 & 25 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 & 0 \end{bmatrix}$$

Enter the following code in Macaulay 2, taken mostly from Hoşten and Smith's "Monomial Ideals" chapter from the Macaulay 2 handbook (edited by Eisenbud et al.).

```
toBinomial = (b,R) -> (
  top := 1_R; bottom := 1_R;
  scan(#b, i -> if b_i > 0 then top = top * R_i^(b_i)
    else if b_i < 0 then bottom = bottom * R_i^(-b_i));
  top - bottom);

toricIdeal = (A, omega) -> (
  n := #(A_0);
  R = QQ[x_1..x_n, Weights => omega, Degrees=>transpose A, MonomialSize=>16];
  B = transpose matrix syz matrix A;
  J := ideal apply(entries B, b -> toBinomial(b,R));
  scan(gens ring J, f -> (J = saturate(J,f)));
  J
);
```

The command `standardPairs` should be included in your Macaulay 2 package but, if not, please see the code (you can copy and paste) at the Monomial Ideals (Hoşten and Smith) chapter at <http://www.math.uiuc.edu/Macaulay2/Book/>.

- (1) Record all the regular subdivisions of **COIN**. Try sketching the secondary fan of **COIN**. *Hint*: two triangulations will neighbour each other if there exists precisely one subdivision that is refined by them and only by them.
- (2) Let  $\mathbf{c} = (3, 0, 0, 1, 0)$ . In Macaulay 2, enter (one line at a time)

```
cee = { 3,0,0,1,0 }
```

along with copying and pasting all the contents at the beginning of this tutorial. Next, find the generating set of  $I_{\mathbf{T}}$ , its Gröbner basis and initial ideal (w.r.t.  $\mathbf{c}$ ) by entering

```
T = {{1,1,1,1,1},{0,1,0,1,2},{0,0,1,1,2}}
```

```
I = toricIdeal (T, cee)
```

```
gens gb I
```

```
M = monomialIdeal I
```

Compare the results with your results from 4ti2 earlier. Finally, find the standard pair decomposition by entering

```
standardPairs M
```

Do you notice the facets of the triangulation  $\Delta_{\mathbf{c}}$  coming from the std. pair decomposition ?

- (3) Repeat the above exercise for  $\mathbf{c} = (3, 0, 0, -1, 0)$ . **Be careful:** You will have to let `cee = { 4,1,1,0,1 }` since the `toricIdeal` command will not permit weight vectors with negative entries; adding any vector in the row space of  $\mathbf{T}$  to a weight vector does not change the linear or integer program so  $(4, 1, 1, 0, 1) = (3, 0, 0, -1, 0) + (1, 1, 1, 1, 1)$ . You can confirm this as an exercise or take it on faith by checking that  $\Delta_{(4,1,1,0,1)}$  equals  $\Delta_{(3,0,0,-1,0)}$ .

How do the standard pairs reflect the fact that the triangulation is unimodular ?

- (4) Study the standard pair decomposition of initial ideals of the matrix  $\mathbf{B}$ .

```
cee2 = {10,5,0,2,6,7,1}
```

```
B = {{1, 1, 1, 1, 1, 1, 1},
```

```
{1, 2, 1, 3, 2, 1, 0},
```

```
{0, 1, 0, 1, 0, 1, 0},
```

```
{5, 4, 3, 2, 1, 0, 0}}
```

```
J = toricIdeal (B, cee2)
```

```
L = monomialIdeal J
```

```
standardPairs L
```

Can you see evidence to support the chain theorem ?

- (5) Using the chain theorem construct a monomial ideal that is not a toric initial ideal.  
 (6) For each of the standard pair decompositions above identify a group relaxation that minimally solves an integer program, in the sense of Page 15 of Lecture II.