Toric Geometry

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Goal:

Compare many of the definitions of toric varieties

Themes:

Toric varieties are a way to bring combinatorics into algebraic geometry.

They are natural generalizations of projective space.

Conventions:

If I is an ideal in the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$, then the affine variety $V(I)$ is

$$V(I) = \{\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n : f(\mathbf{a}) = 0 \text{ for all } f \in I\}.$$

The coordinate ring of $V(I)$ is S/I.

If I is a homogeneous ideal in the polynomial ideal $S = \mathbb{C}[x_0, \ldots, x_n]$, then the projective variety $V(I)$ is

$$V(I) = \{\mathbf{b} = (b_0 : \cdots : b_n) \in \mathbb{P}^n : f(\mathbf{b}) = 0 \text{ for all} f \in I\}.$$

The homogeneous coordinate ring of $V(I)$ is S/I.

Definition: $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The algebraic torus $T = (\mathbb{C}^*)^d$.

Part 1 :

Seven definitions of toric varieties

Definition 1: A toric variety X is a (normal) variety containing a dense copy of the torus $T = (\mathbb{C}^*)^d$ with an action of T on X extending the action of T on itself.

Example: $\mathbb{P}^2 = \{(x_0 : x_1 : x_2) : x_i \in \mathbb{C}\}$.

$T = \{(x_0 : x_1 : x_2) : x_i \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^2$.

(since $(x_0 : x_1 : x_2) = (1 : x_1/x_0 : x_2/x_0)$).

T acts on $\mathbb{P}^2$ by coordinatewise multiplication.

$(t_0 : t_1 : t_2) \cdot (x_0 : x_1 : x_2) = (t_0 x_1 : t_1 x_1 : t_2 x_2)$.

Definition summary:

1. (Normal) variety X with dense torus T and action of T on X.

Definition 2: An affine (respectively projective) toric variety is the closure of the image of a monomial map

$$\phi : T \to \mathbb{C}^d \text{ (respectively } \mathbb{P}^d\text{)}.$$

Example:

$$\begin{aligned} &\phi\colon (\mathbb{C}^*)^2 \longrightarrow \mathbb{C}^4 \\ &(t_1, t_2) \mapsto (t_1, t_1t_2, t_1t_2^2, t_1t_2^3) \end{aligned}$$

This is the variety $V = V(x_1x_4 - x_2x_3, x_1x_3 - x_2^2, x_2x_4 - x_3^2)$.

These equations can be computed naively using elimination (see [CLO, Chapter 3] or [GBCP]).

Example: (continued)

$V = V(x_1x_4 - x_2x_3, x_1x_3 - x_2^2, x_2x_4 - x_3^2) \subset \mathbb{C}^4$ is an affine toric variety.

Check: $V \cap (\mathbb{C}^*)^4 \cong (\mathbb{C}^*)^2$. Fix x_1, x_2. Then $x_3 = x_2^2/x_1$, and $x_4 = x_3^2/x_2 = x_2^3/x_1^2$.

T acts on V by coordinatewise multiplication: $(t_1, t_2) \cdot (x_1, x_2, x_3, x_4) = (t_1x_1, t_2x_2, t_2^2/t_1x_3, t_2^3/t_1^2x_4)$. Check: this lies in V, as $(t_1x_1)(t_2^3/t_1^2x_4) - (t_2x_2)(t_2^2/t_1x_3) = t_2^3/t_1(x_1x_4 - x_2x_3) = 0$.

Definition summary:

1. (Normal) variety X with dense torus T and action of T on X.
2. Closure of image of monomial map.

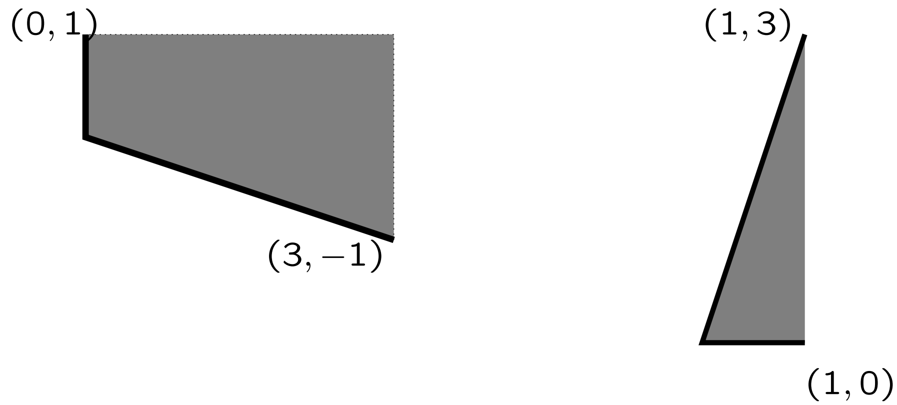
Definition 3: An affine toric variety is a variety $V(I)$ where $I \subset S := \mathbb{C}[x_1, \ldots, x_n]$ and S/I is a semigroup algebra (equivalently $U_\sigma = \mathrm{Spec}(\mathbb{C}[R])$ where R is a semigroup algebra).

In the normal case:

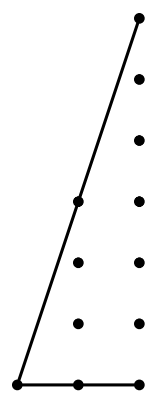
Let $\sigma \subset \mathbb{R}^d$ be a (rational, strongly convex) polyhedral cone, and let $\sigma^\vee = \{x \in \mathbb{R}^d : x \cdot y \geq 0 \text{ for all } y \in \sigma\}$.

Let $S_\sigma = \sigma^\vee \cap \mathbb{Z}^d$. This is a finitely generated semigroup. The coordinate ring of U_σ is $\mathbb{C}[S_\sigma] = \mathbb{C}[t^u : u \in S_\sigma] \subset \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$, where $t^u = t_1^{u_1} t_2^{u_2} \ldots t_d^{u_d}$.

Example:



$$\sigma^\vee = \{x \in \mathbb{R}^d : x \cdot y \geq 0 \text{ for all } y \in \sigma\}$$



So S_σ is generated by

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}.$$

Thus $\mathbb{C}[S_\sigma] = \mathbb{C}[t_1, t_1t_2, t_1t_2^2, t_1t_2^3]$.

So $\mathbb{C}[S_\sigma] \cong \mathbb{C}[x_1, x_2, x_3, x_4]/\langle x_1x_4 - x_2x_3, x_1x_3 - x_2^2, x_2x_4 - x_3^2\rangle$, and so the corresponding toric variety is $V(x_1x_4 - x_2x_3, x_1x_3 - x_2^2, x_2x_4 - x_3^2)$ (the affine cone over the twisted cubic curve).

This is the closure of the image of the map $\phi : (\mathbb{C}^*)^2 \to \mathbb{C}^4$ given by $\phi : (t_1, t_2) \mapsto (t_1, t_1t_2, t_1t_2^2, t_1t_2^3)$.

Let A be the $d \times n$ matrix whose columns are the generators for S_σ.

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

Then the toric ideal I_A is

$$I_A = \langle x^u - x^v : u, v \in \mathbb{N}^n, Au = Av \rangle,$$

where $x^u = x_1^{u_1} x_2^{u_2} \dots x_n^{u_n}$.

Example: $I_A = \langle x_1 x_4 - x_2 x_3, x_1 x_3 - x_2^2, x_2 x_4 - x_3^2 \rangle$.

The affine toric variety is $V(I_A)$.

Definition summary:

1. (Normal) variety X with dense torus T and action of T on X.
2. Closure of image of monomial map.
3. Affine variety with coordinate ring a semigroup ring.

Definition 4: A binomial ideal has generators

$$I = \langle x^{\mathbf{u}_1} - x^{\mathbf{v}_1}, \ldots, x^{\mathbf{u}_s} - x^{\mathbf{v}_s} \rangle \subset \mathbb{C}[x_1, \ldots, x_n],$$

where $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{N}^n$ for $1 \leq i \leq s$.

When I is prime, then I is a toric ideal ($I = I_A$ for some matrix A), so $V(I) \subset \mathbb{C}^d$ is an affine toric variety (or projective, if I is a homogeneous ideal).

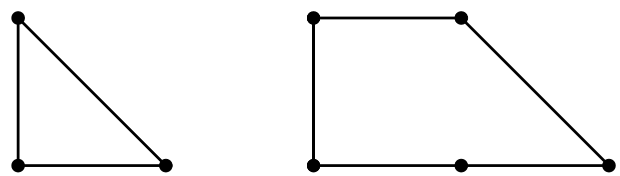
For general binomial I each irreducible component of $V(I)$ is a toric variety. (†) [Eisenbud-Sturmfels]

Definition summary:

1. (Normal) variety X with dense torus T and action of T on X.
2. Closure of image of monomial map.
3. Affine variety with coordinate ring a semigroup ring.
4. Variety of prime binomial ideal.

Definition 5: A projective toric variety X_P is determined by a lattice polytope $P \subset \mathbb{R}^d$.

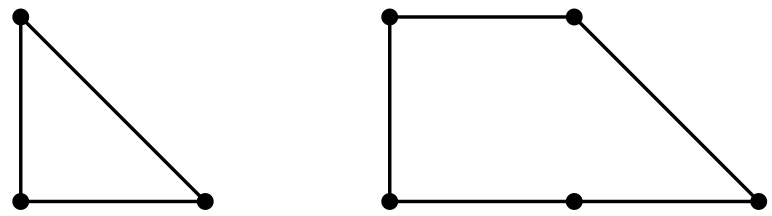
Example:



The toric variety X_P has homogeneous coordinate ring

$$\mathbb{C}[P] = k[t^u s^k : u \in kP \cap \mathbb{Z}^d] \subset \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, s],$$

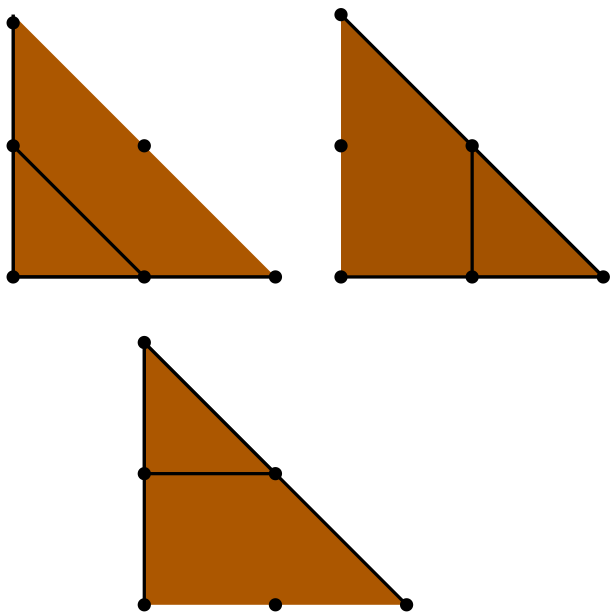
where the grading is $\deg(s) = 1, \deg(t_i) = 0$ for $1 \leq i \leq d$. This is generated in degree one in nice situations.



Example: $\mathbb{C}[P] = k[s, t_1 s, t_2 s] \cong \mathbb{C}[x_1, x_2, x_3]$. (So X_P is $\mathbb{P}^2$).

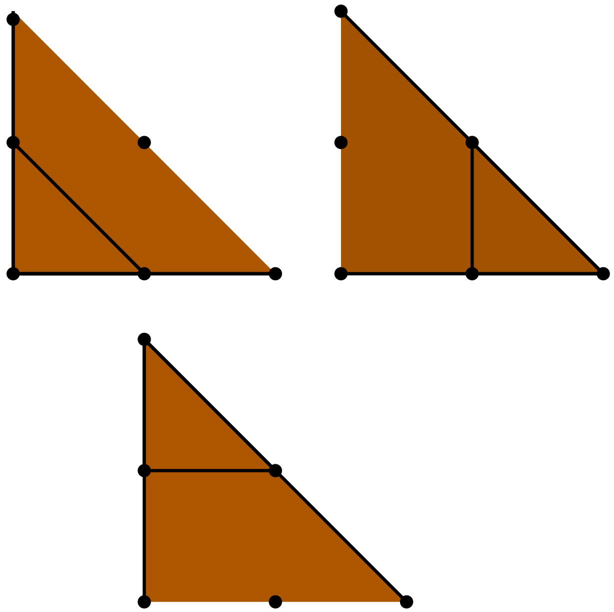
Example: $\mathbb{C}[P] = \mathbb{C}[s, t_1 s, t_1^2 s, t_2 s, t_1 t_2 s] \cong \mathbb{C}[x_1, x_2, x_3, x_4, x_5]/\langle x_3 x_4 - x_2 x_5, x_2 x_4 - x_1 x_5, x_2^2 - x_1 x_3 \rangle$. (Here X_P is the Hirzebruch surface $\mathbb{F}_1$).

The variety X_P has an affine cover given by affine toric varieties corresponding to the vertices of the polytope.



Example: $\mathbb{C}[t_1, t_2]$, $\mathbb{C}[t_1^{-1}, t_1^{-1}t_2]$, and $\mathbb{C}[t_1 t_2^{-1}, t_2^{-1}]$ are the three coordinate rings. These all describe $\mathbb{C}^2$.

Example: (continued)

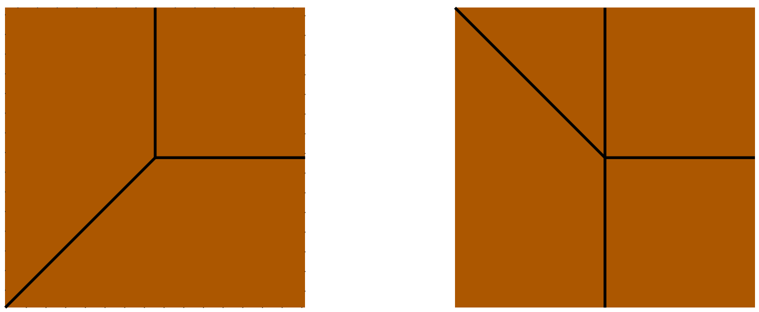


The three coordinate rings $\mathbb{C}[t_1, t_2]$, $\mathbb{C}[t_1^{-1}, t_1^{-1}t_2]$, and $\mathbb{C}[t_1t_2^{-1}, t_2^{-1}]$ are the familiar affine cover of $\mathbb{P}^2$: $\mathbb{C}[x_1/x_0, x_2/x_0]$, $\mathbb{C}[x_0/x_1, x_2/x_1]$, $\mathbb{C}[x_0/x_2, x_1/x_2]$, setting $t_1 = x_1/x_0$, and $t_2 = x_2/x_0$.

Definition summary:

1. (Normal) variety X with dense torus T and action of T on X.
2. Closure of image of monomial map.
3. Affine variety with coordinate ring a semigroup ring.
4. Variety of prime binomial ideal.
5. Variety determined by polytope.

Definition 6 : A rational polyhedral fan Σ is a collection of rational polyhedral cones, with the intersection of any two being a face of each.

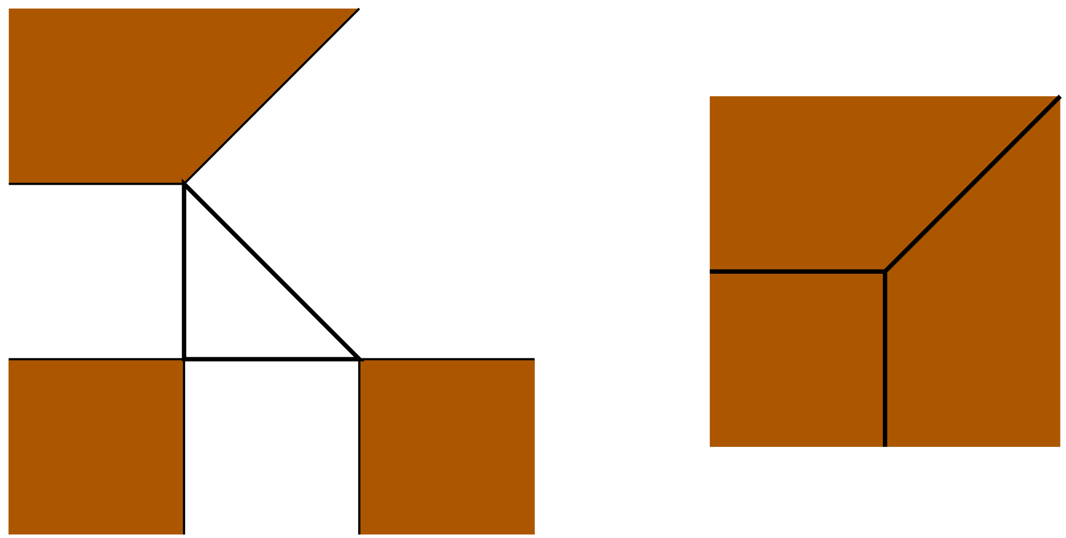


The corresponding toric variety X_Σ is obtained by gluing together the affine toric varieties U_σ corresponding to cones $\sigma \in \Sigma$ along subvarieties corresponding to the common faces.

Example: $\mathbb{P}^2$ is obtained by gluing together three copies of $\mathbb{C}^2$ along copies of $\mathbb{C} \times \mathbb{C}^*$. ($\mathbb{P}^2 = \{(x : y : z) : x, y, z \in \mathbb{C}\}$, and the three copies of $\mathbb{C}^2$ are $x \neq 0$, $y \neq 0$, and $z \neq 0$.)

Example: $\mathbb{F}_1$ is obtained by gluing together four copies of $\mathbb{C}^2$ along copies of $\mathbb{C} \times \mathbb{C}^*$.

Every convex polytope P has a normal fan $\mathcal{N}(P)$.



Then $X_{\mathcal{N}(P)} = X_P$.

Definition summary:

1. (Normal) variety X with dense torus T and action of T on X.
2. Closure of image of monomial map.
3. Affine variety with coordinate ring a semigroup ring.
4. Variety of prime binomial ideal.
5. Variety determined by polytope.
6. Variety determined by fan.

Definition 7: A toric variety of dimension $(n-d)$ is a GIT quotient of $\mathbb{C}^n$ for some n by a d-dimensional torus.

Concretely, let the d-dimensional torus H act on $\mathbb{C}^n$ by

$$(t \cdot x)_i = t^{\mathbf{a}_i} x_i,$$

where $\mathbf{a}_i \in \mathbb{Z}^d$, and $t = (t_1, \ldots, t_d)$.

Given $p \in \mathbb{R}^d$ generic, let $B = \langle \prod_{i \in \sigma} x_i : p \in \text{pos}(\mathbf{a}_i : i \in \sigma) \rangle$, and let $V = V(B)$.

Then $X = (\mathbb{C}^n \setminus V)/H$ is a toric variety with torus $(\mathbb{C}^*)^n/H$.

Example:

$\mathbb{C}^*$ acts on $\mathbb{C}^3$ by $t \cdot (x_1, x_2, x_3) = (tx_1, tx_2, tx_3)$.

Then for $p = 1 \in \mathbb{R}$, we get $B = \langle x_1, x_2, x_3 \rangle$, so $V(B) = (0, 0, 0)$.

Then $(\mathbb{C}^3 \setminus (0, 0, 0))/\mathbb{C}^* = \mathbb{P}^2$.

Example:

$(\mathbb{C}^*)^2$ acts on $\mathbb{C}^4$ by $(t_1, t_2) \cdot (x_1, x_2, x_3, x_4) = (t_1 x_1, t_2/t_1 x_2, t_1 x_3, t_2 x_4)$. For $p = (1, 1)$ we get $B = \langle x_1 x_2, x_2 x_3, x_3 x_4, x_1 x_4 \rangle$, so $V(B)$ is a collection of four 2-planes in $\mathbb{R}^4$. Then $(\mathbb{C}^4 \setminus V(B))/(\mathbb{C}^*)^2 = \mathbb{F}_1$.

How do these definitions all compare?

1. (Normal) variety X with dense torus T and action of T on X.
2. Closure of image of monomial map.
3. Affine variety with coordinate ring a semigroup ring.
4. Variety of prime binomial ideal.
5. Variety determined by polytope.
6. Variety determined by fan.
7. Quotient of affine space.

In the normal case, 1 = 6 is the most general defintion, and 3 and 5 are special subcases, with 7 a special case of 5. There is also a more general form of 7 that equals 1=6.

How do these definitions all compare?

1. (Normal) variety X with dense torus T and action of T on X.
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6. Variety determined by fan.
7. Quotient of affine space.

Definition 1,2,3,4 have nonnormal versions. Then 2=3=4 in the affine case, and 2=4 in the projective case; again 1 is the most general definition.

Part 2: Uses of toric varieties

One use of toric varieties is to test conjectures in algebraic geometry.

Many properties of the variety can be determined by the combinatorics of the defining data, so a conjecture about varieties can be turned into a combinatorial problem that may be easier to check.

Example: The dimension of a projective toric variety is the dimension of the defining polytope.

Example: An affine toric variety given by a cone σ is smooth if and only if σ is isomorphic to cone generated by some of the standard basis vectors e_i for $\mathbb{R}^n$. Here "isomorphic" means that there is a matrix in $\mathrm{GL}(n, \mathbb{Z})$ that takes one cone to the other.

The toric variety X determined by a fan Σ is smooth if and only if each cone $\sigma \in \Sigma$ is isomorphic to one generated by some of the standard basis vectors.

There is a combinatorial algorithm to resolve singularities of toric varieties.

Many other properties of a toric variety X have nice combinatorial descriptions, such as the cohomology ring, the divisor class group, Chow ring, nef and ample cones, etc.

Smooth toric varieties are a natural generalization of projective space.

The second main use of toric varieties is as a nice ambient variety in which to study other varieties. Instead of embedding a variety into projective space, we can embed it into a toric variety.

If $X \subset \mathbb{P}^N$, it is often possible to embed X into a toric variety of dimension less than N, and the combinatorics of the toric variety may better reflect the properties of X.

This can be described algebraically using the homogeneous coordinate ring of a toric variety (also known as the Cox ring).

Classical case: Projective space. Let $S = \mathbb{C}[x_0, \ldots, x_n]$, and $\mathfrak{m} = \langle x_0, \ldots, x_n \rangle$.

Subvarieties of $\mathbb{P}^n$ correspond to homogeneous, radical ideals in S that are saturated with respect to $\mathfrak{m}$.

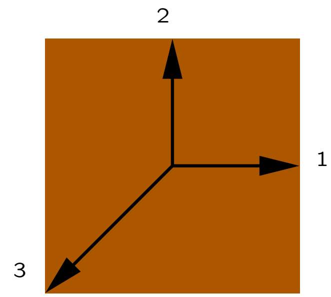
Subschemes of $\mathbb{P}^n$ correspond to homogeneous ideals in S that are saturated with respect to $\mathfrak{m}$.

Coherent sheaves on $\mathbb{P}^n$ correspond to graded S-modules (modulo an equivalence relation).

Smooth toric case: Let X be a smooth toric variety with fan Σ. Let $S = \mathbb{C}[x_1, \ldots, x_r]$, where r is the number of rays of Σ. We grade S by the Divisor class group of X. This is is $\mathbb{Z}^r / \operatorname{im}(R)$, where R is the $(r \times d)$-matrix whose rows are the first lattice point on each ray.

Let $B = \langle \prod_{i \notin \sigma} x_i : \sigma \in \Sigma \rangle$. This is the irrelevant ideal of S.

Example:



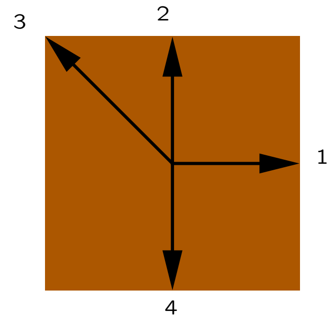
Σ has three rays, so $S = \mathbb{C}[x_1, x_2, x_3]$.

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \text{ and } \mathbb{Z}^3/\operatorname{im}(R) \cong \mathbb{Z},$$

which gives $\deg(x_i) = 1$ for $i = 1, 2, 3$. $B = \langle x_1, x_2, x_3 \rangle$.

This is (after renumbering) the standard coordinate ring of $\mathbb{P}^2$.

Example:



Σ has four rays, so $S = \mathbb{C}[x_1, x_2, x_3, x_4]$.

$$R = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}^T, \text{ and } \mathbb{Z}^4/\operatorname{im}(R) \cong \mathbb{Z}^2,$$

with $\deg(x_1) = \deg(x_3) = (1, 0)$, $\deg(x_2) = (-1, 1)$, and $\deg(x_4) = (0, 1)$. $B = \langle x_1x_2, x_2x_3, x_3x_4, x_1x_4 \rangle$.

Classical case: Subvarieties of $\mathbb{P}^n$ correspond to homogeneous, radical ideals in S that are saturated with respect to $\mathfrak{m}$.

Subschemes of $\mathbb{P}^n$ correspond to homogeneous ideals in S that are saturated with respect to $\mathfrak{m}$.

Coherent sheaves on $\mathbb{P}^n$ correspond to graded S-modules (modulo an equivalence relation).

Smooth toric case: Let X be a smooth toric variety.

Subvarieties of X correspond to homogeneous, radical ideals in S that are saturated with respect to B.

Subschemes of X correspond to homogeneous ideals in S that are saturated with respect to B.

Coherent sheaves on X correspond to graded S-modules (modulo an equivalence relation).

Brings commutative algebra into toric geometry!

Summary: Toric varieties have many related combinatorial descriptions.

Many geometric properties of toric varieties have nice combinatorial descriptions.

Toric varieties are generalizations of projective space, and are a good choice of ambient space to embed other varieties.

Where to learn more?

1. Fulton *Introduction to toric varieties* (see also Mustaţă's notes).
2. Oda *Convex bodies and algebraic geometry*, Danilov *The geometry of toric varieties*.
3. Ewald *Combinatorial convexity and algebraic geometry*.
4. Sturmfels *Gröbner bases and convex polytopes*.
5. Cox-Little-Schenck (forthcoming)