Stability Region Analysis Using Simulations and Sum-of-Squares Programming

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Estimating the Region-of-Attraction

Dynamics with equilibrium point at $\bar{x}$

$$\dot{x} = f(x), \quad f(\bar{x}) = 0$$

User-defined function p, whose sublevel sets are contained in the region-of-attraction

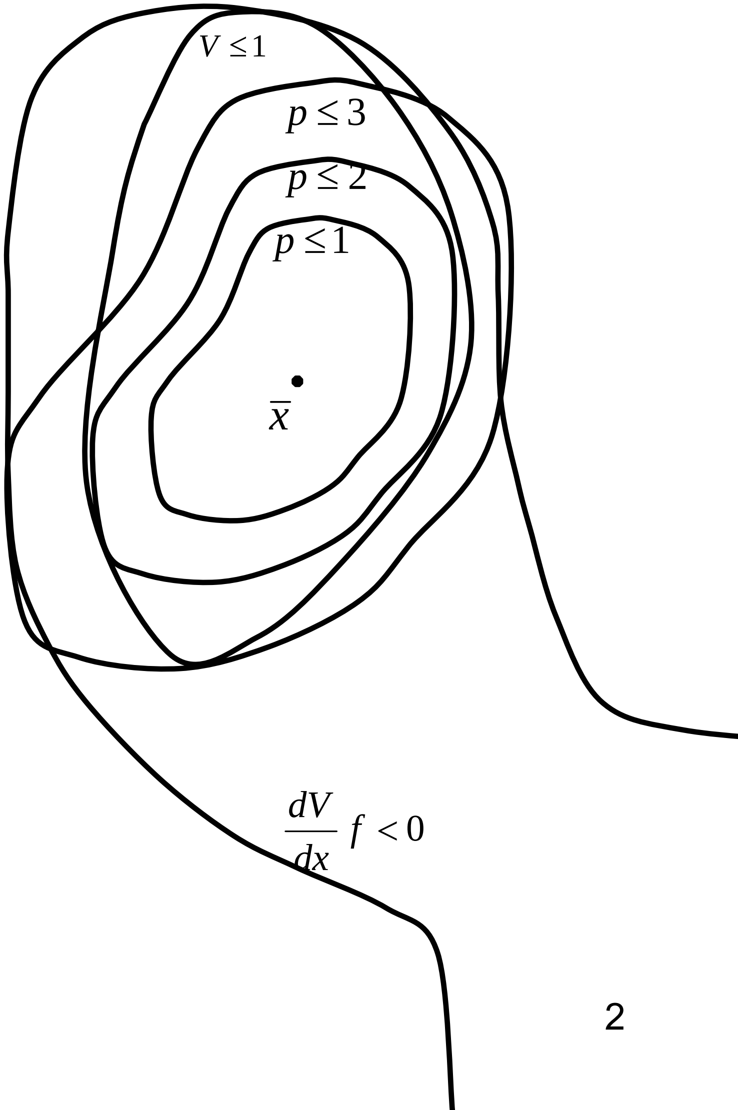
$$P_\beta := \{x : p(x) \le \beta\} \subseteq ROA_{\bar{x}}$$

By choice of positive definite V (Lyapunov function) maximize β such that

$$\{x : V(x) \le 1\} \text{ is bounded}$$

$$\{x : p(x) \le \beta\} \subseteq \{x : V(x) \le 1\}$$

$$\{x : x \ne \bar{x}, V(x) \le 1\} \subseteq \left\{x : \frac{dV}{dx} f < 0\right\}$$



Nonconvexity of Local Analysis

Local analysis based on Lyapunov functions involve search over nonconvex sets of Lyapunov function coefficients due to constraints in the form

$$\{x : x \neq \bar{x}, V(x) \leq 1\} \subseteq \left\{ x : \frac{dV}{dx} f < 0 \right\}$$

Nonconvex:

$$f(x) = -x$$

$$V_1(x) = 16x^2 - 19.95x^3 + 6.4x^4$$

$$V_2(x) = 0.1x^2$$

$$V_c(x) = 0.58 \cdot V_1(x) + 0.42 \cdot V_2(x)$$

V_1 and V_2 satisfy above constraint, but V_c (a convex combination of V_1 and V_2) does not.

Sum-of-Squares Relaxation: Nonconvex

Sum-of-Squares relaxation for

$$\{x : x \neq \bar{x}, V(x) \leq 1\} \subseteq \left\{x : \frac{dV}{dx} f < 0\right\}$$

using the generalized S-procedure

$$\mathit{find} \quad V, V(0) = 0, \mathit{and}\ s_2, s_3 \in \Sigma_n$$

subject to

$$-\left((1 - V)s_2 + s_3 \nabla V \cdot f + l_2\right) \in \Sigma_n$$

Bilinear SDP (SDP with BMI constraint), usually solved by

- Ad-hoc iterative linear schemes
- Local bilinear search (such as PENBMI).

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Behavior:

– Initial point has big effect on end result, e.g.,
  - Unable to reach a feasible point
  - Convergence to local optimum

What are prospects for generating “good” initial points?

– Easily computable

– Promising results

Region of Attraction

Consider a simpler question. Fix *β*, is

$$P_\beta := \{x : p(x) \le \beta\} \subseteq ROA_{\bar{x}} \ ?$$

Ad-hoc solution:

- run N sims, starting from samples in $\{x : p(x) \le \beta\}$
  - If any diverge, then "no"
  - If all converge, then maybe "yes", and perhaps the Lyapunov analysis can prove it

In this case, how can we use the simulation data?

Necessary condition: If *V* exists to verify, it **must** be

- ≤1 on all trajectories
- ≥0 on all trajectories
- Decreasing on all trajectories
- and possibly some more…

$$\{x : p(x) \le \beta\} \subseteq \{x : V(x) \le 1\}_{bounded}$$

$$\{x : x \ne \bar{x}, V(x) \le 1\} \subseteq \left\{x : \frac{dV}{dx} f < 0\right\}$$

Outer bound on certifying Lyapunov functions

After simulations

- Collection of convergent trajectories starting in P_β
- divergent trajectories starting in P_β^c

Linearly parametrize *V*, namely

$$V(x) = \sum_{k=1}^{N_b} \alpha_k \phi_k(x)$$

The necessary conditions on V are convex constraints on

V≤1 on convergent trajectories

V≥0 on all trajectories

$$V(x) - l_1(x) \in \Sigma_n(x)$$

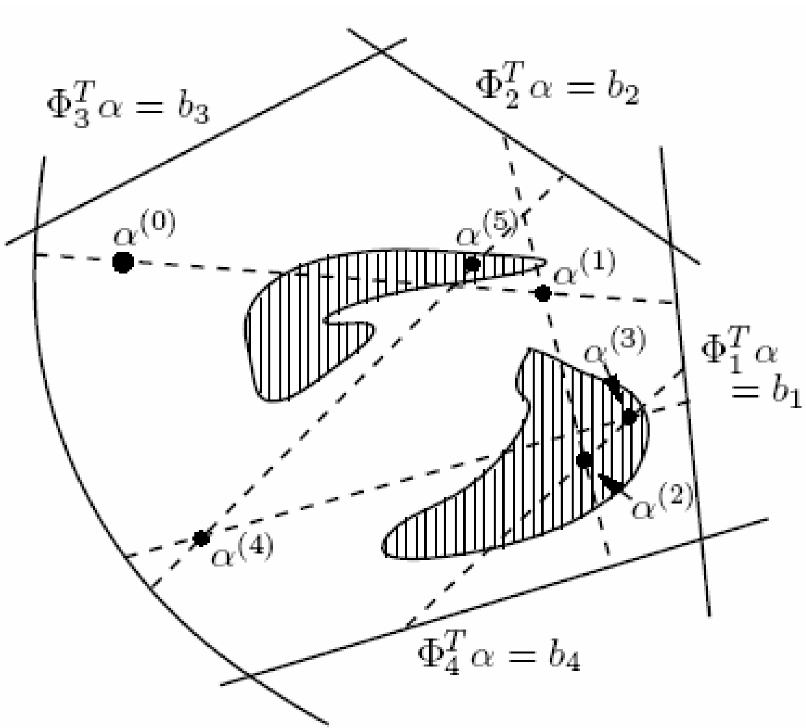
V decreasing on convergent trajectories

Quad(V) is a Lyapunov function for Linear(f)

V≥1 on divergent trajectories

Hit & Run: Uniformly sample convex set in R^n *to Generate Lyapunov Function Candidates*

1. Start with an interior point, *w*
2. Pick a direction *v* in R^n, *N(0,I)*
3. Find t_{min} and t_{max} such that *w+tv* just in set
4. Pick *μ*, uniformly in $[t_{min}\ t_{max}]$
5. Next x = x + *μv*



In Lyapunov coefficient space, get samples:

- Assess the ROA that they certify, or…
- Use as a seed for
  - PENBMI, and/or iteration

Finding $[t_{min}\ t_{max}]$ involves

- Several simple 1-d linear inequalities
- A linear matrix inequality for

$$A^T P + PA \prec 0$$

- An SOS program, for

$$V - l_1 \in \Sigma_n$$

Smith, 1984 *Operations Research*
Lovasz, 1999 *Math Programming*
Tempo, Calafiore, Dabbene, *Springer*

Assessing V: Checking containments

Each candidate V certifies a region-of-attraction

$$\beta_{cert,V} := \max \beta \ \ such\ that\ \ \exists \gamma \ satisfying$$

$$\{x : p(x) \le \beta\} \subseteq \{x : V(x) \le \gamma\} \subseteq \{x : \dot{V}(x) < 0\}$$

Generally, this is solved in two steps

- SOS optimization (s_1, s_2) to maximize the level-set condition on V

$$-\left((\gamma - V)s_1 + s_2 \nabla V \cdot f + l_2\right) \in \Sigma_n$$

- SOS optimization (s_3) to maximize the condition on p & V

$$-\left((\beta - p)s_3 + (V - \gamma)\right) \in \Sigma_n$$

PENBMI and iteration initialized with these as well

Assessing V: Checking containments

Alternate conditions, this is solved in two steps

– SOS optimization (p_1) to maximize the level-set condition on V

$$(V-\gamma)\sum_{i=1}^{n} x_i^{2d_1} - p_1 \nabla V \cdot f \in \Sigma_n$$ SDP, no bisection

– SOS optimization (p_2) to maximize β under the condition on p & V

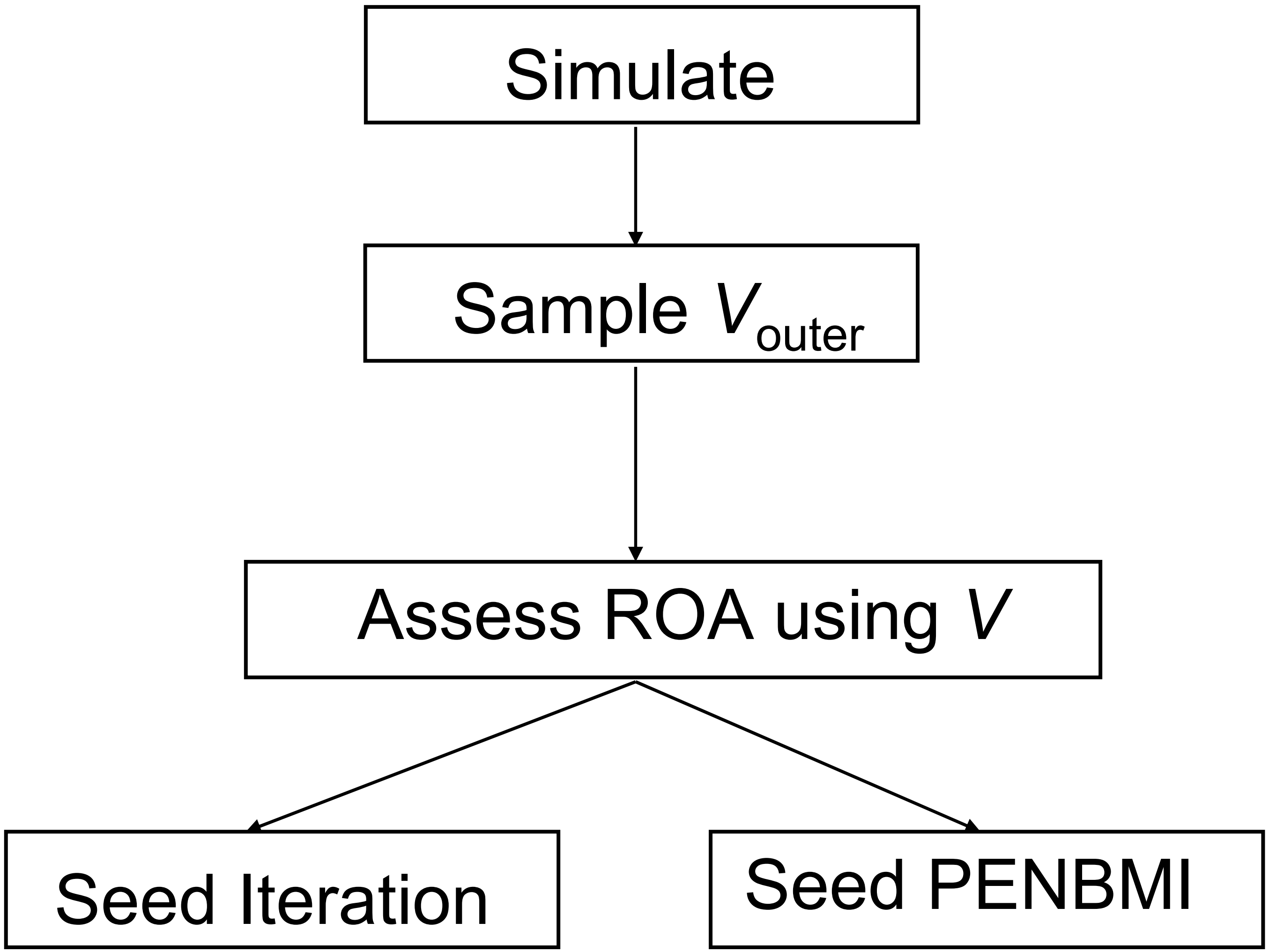
$$-(\beta - p)\sum_{i=1}^{n} x_i^{2d_2} - (V-\gamma)p_2 \in \Sigma_n$$ SDP, no bisection

Under the assumption that $\nabla V \cdot f$ is negative definite near 0, these confirm

$$\beta_{cert,V} := \max \beta \ \ \textit{such that} \ \exists \gamma \ \textit{satisfying}$$

$$\{x : p(x) \le \beta\} \subseteq \{x : x \ne 0, V(x) < \gamma\}_{cc,0} \subseteq \{x : \dot{V}(x) < 0\}$$

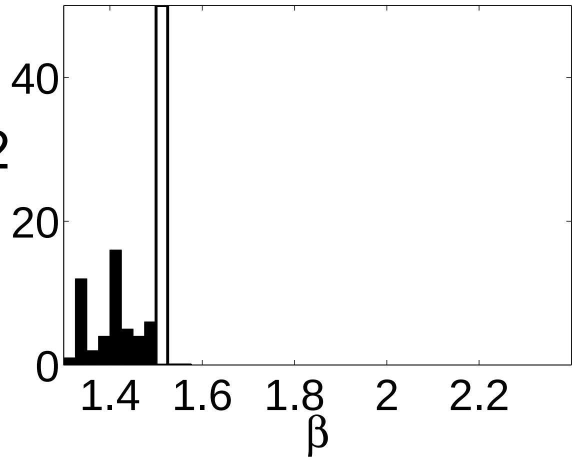
Summary of the Methodology



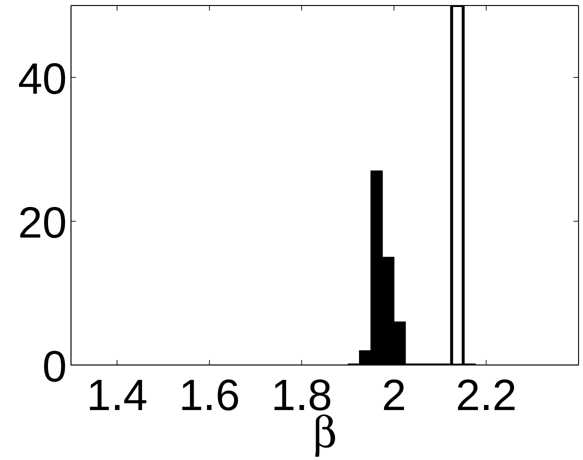
Results: Van der Pol Dynamics

Histogram of the values of β assessing V’s from the sample (black bar) and from further PENBMI runs (white bar).

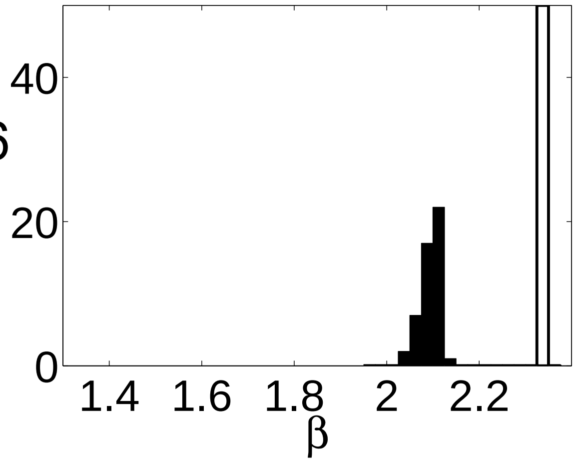
deg(V)=2

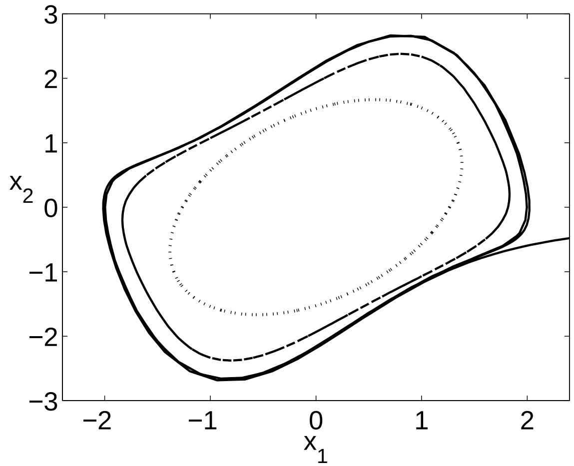


deg(V)=4



deg(V)=6





Van der Pol's Summary

best β	deg(V)=2	deg(V)=4	deg(V)=6
V's from sampling	1.50	2.02	2.13
PENBMI runs	1.52	2.14	2.34

	deg(V)=4	deg(V)=6
time for simulations, linear problem, and generating candidate V's	20-25 sec	40-50 sec
seeded PENBMI run time (50 runs)	3-8 sec	11-24 sec
unseeded PENBMI run time (10 runs)	50-200 sec	1000-2000 sec
seeded PENBMI success ratio	1.0	1.0
unseeded PENBMI success ratio	0.9	0.5

Controlled Short Period Aircraft Dynamics

Aircraft: short period dynamics with LTI dynamic inversion based controller

$$\dot{x} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & 0 & \bullet & \bullet & \bullet \\ \bullet & 0 & 0 & 0 & \bullet \\ \bullet & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} x_1 x_2 + x_2^2 + x_2 x_3 + x_2 x_4 + x_2 x_5 + x_2^3 \\ x_2 x_5 + x_5^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} q \\ \alpha \\ \eta_1 \\ \eta_2 \\ \theta \end{bmatrix}$$

Several Analysis

- Unseeded calls to PENBMI
- Sim-based initialization for iteration
  - Did not run a seeded PENBMI
- Alternate initialization for iteration
- Separate extensive simulations to find divergent trajectories

$$\max \beta \quad \textit{over} \quad V, V(\bar{x}) = 0, s_1, s_2, s_3 \in \Sigma_n$$

subject to

$$V - l_1 \in \Sigma_n$$
$$-\big((\beta - p)s_3 + (V - 1)\big) \in \Sigma_n$$
$$-\big((1 - V)s_1 + s_2 \nabla V \cdot f + l_2\big) \in \Sigma_n$$

Results: Aircraft Dynamics with Quartic *V*

There is a divergent trajectory starting from $p(x_0) = 16.1$

Simulation-based algorithm

4000 simulations	5 minutes → divergent trajectory from $p(x_0) = 16.9$
Form LP/ConvexP	3 minutes
Get a feasible point	5 minutes
Assess answer with V	2 minutes
Iterate from V	3 minutes/iteration, 6 iters → $\{x : p(x) \leq 14.6\} \subseteq ROA$
TOTAL	33 minutes

Iteration from "random" starting point

Take P from $A^T P + PA = -I$

$$V(x) = x^T Px + 0.001\sum_{i=1}^{5} x_i^4$$

→ 30 iterations

$\{x : p(x) \leq 8.5\} \subseteq ROA$

Direct unseeded call to PENBMI yields (after 38 hours)

– All initial conditions in $\{x : p(x) \leq 15.2\}$ are in ROA

What's possible?

Assuming no breakthroughs in

- SDP/BMI solvers
- exploiting problem structure

Then, reliable and time-tolerable analysis for systems with

- Cubic vector fields
- State dimension between 10 and 15, pointwise-max quadratic Lyapunov functions
- State dimension ≤6, quartic Lyapunov functions

How should this be viewed?

- Linearized analysis is effectively
  - Infinitesimal analysis of dynamics with quadratic Lyapunov fcns
- So, the proposed method extends both the degree of approximation of the dynamics, and the richness of the Lyapunov function