Sensor Network Localization, Euclidean Distance Matrix Completions, and Graph Realization

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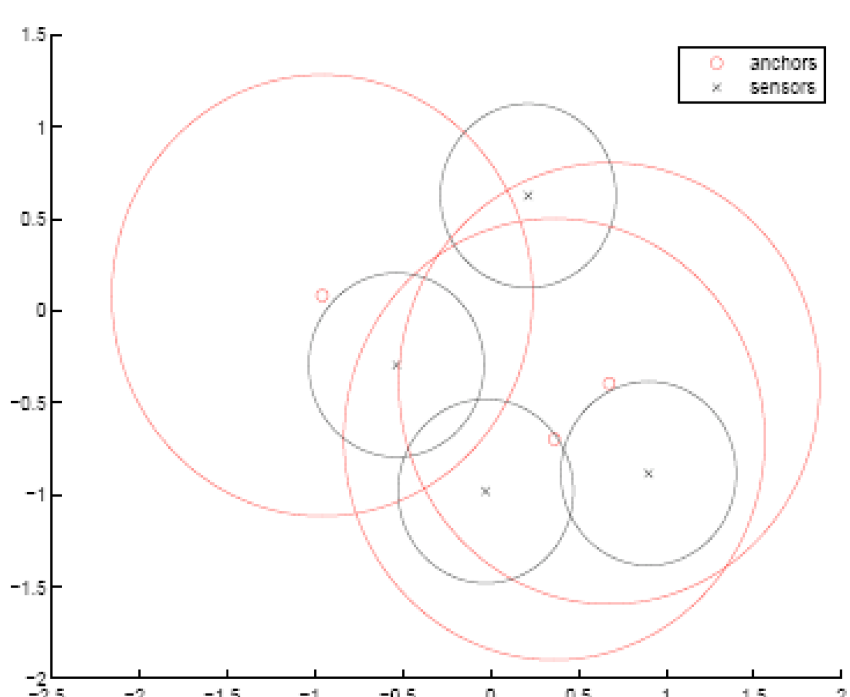
Problem:

Many applications use ad hoc wireless sensor networks for monitoring information. Typical networks include a large number of sensor nodes which gather data and communicate among themselves. The location of a subset of the sensors is known; these sensors are called anchors. From the intercommunication, we are able to establish distances between a subset of the sensors and anchors. The sensor localization problem is to find/estimate the location of all the sensors.

$$\begin{array}{ll} \min & \frac{1}{2}\sum W_{ij}(\|p^i-p^j\|^2-E_{ij})^2 \\ & +\frac{1}{2}\sum W_{ik}(\|p^i-a^k\|^2-E_{ik})^2 \\ \text{s.t.} & \|p^i-p^j\|^2 \le U_{ij} \\ & \|p^i-a^k\|^2 \le U_{ik} \\ & \|p^i-p^j\|^2 \ge L_{ij} \\ & \|p^i-a^k\|^2 \ge L_{ik} \end{array}$$

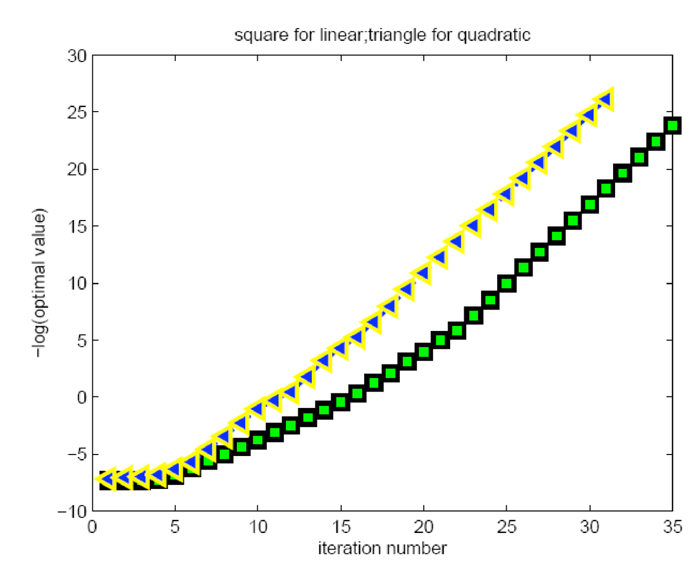


$p^1,\dots,p^n \in \Re^r$	unknown (sensor) points
$a^1,\dots,a^m \in \Re^r$	known (anchor) points
r	embedding dimension (usually 2 or 3)
W_p, W_a	weight matrices
E	partial matrix of known squared distances
U, L	partial matrices of upper and lower bounds

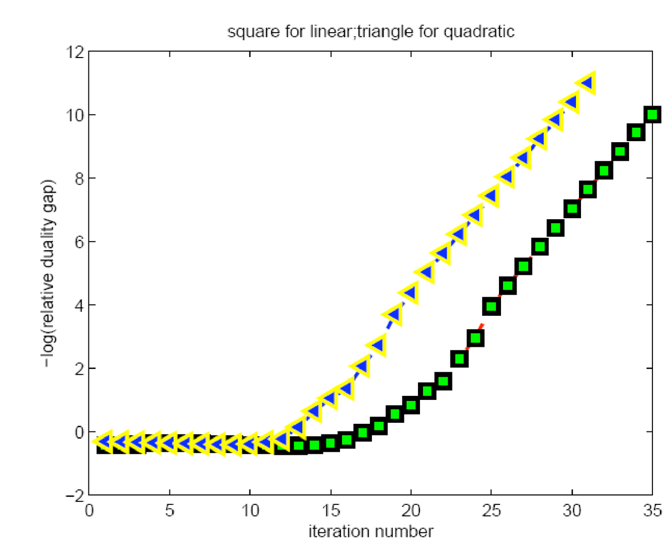
Applications:

- health, military, home
- natural habitat monitor, earthquake detection, weather/current monitoring
- random deployment in inaccessible terrains or disaster relief operations

Results:



Comparison for two barriers, Optimal Value



Comparison for two barriers, Relative Gap

	test 1	test 2	test 3	test 4	test 5	test 6	test 7	mean
Method 1	1.2780	1.4200	1.4801	1.3696	1.1820	1.4317	1.3912	1.3647
Method 2	0.1887	0.1630	0.1050	0.1394	0.0778	0.0808	0.3881	0.1633

Use distance of X estimates from true sensor locations X^*

Theory:

SDP Relaxation based on EDM:

$$\begin{array}{rcl} A^T &:=& [a^1, a^2, \dots, a^m] \\ X^T &:=& [p^1, p^2, \dots, p^n] \\ P^T &:=& (X^T \quad A^T) \\ \bar{Y} &:=& PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & AA^T \end{pmatrix} \end{array}$$

$$\begin{array}{ll} \min & \frac{1}{2}\|W\circ(\mathcal{K}(\bar{Y})-E)\|_F^2 \\ \text{s.t.} & H_u\circ(\mathcal{K}(\bar{Y})-U)\le 0 \\ & H_l\circ(\mathcal{K}(\bar{Y})-L)\ge 0 \\ & \bar{Y} = \begin{pmatrix} Y & XA^T \\ AX^T & AA^T \end{pmatrix} \\ & \boxed{Y = XX^T} \qquad \text{(hard constraint)} \end{array}$$

Calculate the distances from the points:

$$\mathcal{K}(\bar{Y})_{ij} := \left(\operatorname{diag}(\bar{Y})e^T + e\operatorname{diag}(\bar{Y})^T - 2\bar{Y}\right)_{ij} = \|p^i-p^j\|^2$$

SDP Relaxation: $Y = XX^T \;\Rightarrow\; Y \succeq XX^T$

To Linearize or Not to Linearize… (two barriers)

$$\begin{pmatrix} I & X^T \\ X & Y \end{pmatrix} \succeq 0 \;\Leftrightarrow\; Y \succeq XX^T$$

+ Linear Constraint	- Quadratic Constraint
- NOT Onto	+ Onto
- Larger Dimension	+ Smaller Dimension

Clique Reductions using Minimal cone Projection:

$$\begin{array}{ll} \min & \frac{1}{2}\|W\circ(\mathcal{K}(\bar{Y})-E)\|_F^2 \\ \text{s.t.} & H_u\circ(\mathcal{K}(\bar{Y})-U)\le 0 \\ & H_l\circ(\mathcal{K}(\bar{Y})-L)\ge 0 \\ & \boxed{\bar{Y} = PP^T} \qquad \text{(hard constraint)} \end{array}$$

SDP Relaxation: $\bar{Y} = PP^T \;\Rightarrow\; \bar{Y}\succeq 0,\ \operatorname{sblk}_2(\bar{Y}) = AA^T$

Why? If $\bar{Y} = \begin{pmatrix} Y_{11} & Y_{21}^T \\ Y_{21} & AA^T \end{pmatrix} \succeq 0$, then $Y_{21}^T = XA^T$, where $X = Y_{21}^T A(A^TA)^{-1}$.

i.e. X not needed in SDP

Since $\operatorname{sblk}_2(\bar{Y}) = AA^T$, $\bar{Y}$ cannot be strictly positive definite, so the Slater constraint qualification fails. Therefore, we project onto the minimal face of the SDP cone that contains the feasible set.

$$\begin{array}{ll} \min & \frac{1}{2}\|W\circ(\mathcal{K}(U_AZU_A^T)-E)\|_F^2 \\ \text{s.t.} & H_u\circ(\mathcal{K}(U_AZU_A^T)-U)\le 0 \\ & H_l\circ(\mathcal{K}(U_AZU_A^T)-L)\ge 0 \\ & \operatorname{sblk}_2(Z) = I \\ & Z\succeq 0 \end{array} \quad \Leftarrow \quad U_A = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$$

Note: $\bar{Y}\in\mathcal{S}^{n+m}$ and $Z\in\mathcal{S}^{n+r}$, $(r<m)$

If we have another clique of $p>r$ sensors (i.e. all distances are known), then we can use a similar technique to reduce the problem even further.

Robust Primal-Dual Interior-Point Method:

Key Steps:

1. Eliminate variables from the dual constraints.
2. Form perturbed optimality conditions.
3. Start with an initial strictly feasible point.
4. Apply Gauss-Newton search direction found by using a preconditioned conjugate gradient method.
5. When close enough to the optimal solution, Cross-over, that is set centering parameter to 0 and take a step of length 1 in each iteration. (Fast convergence to optimal solution)

Estimating Sensor Locations based on EDM Model:

<u>Method 1:</u> Obtain X from the optimal $Z_s = \begin{pmatrix} I & X^T \\ X & Y \end{pmatrix}$, or, equivalently, let $X = Y_{21}^T A(A^TA)^{-1}$, where $\bar{Y} = \begin{pmatrix} Y & Y_{21}^T \\ Y_{21} & AA^T \end{pmatrix}$.

<u>Method 2:</u> Suppose $\bar{Y}_r := U_r\Sigma_rU_r^T$ is the best rank r approximation of the optimal $\bar{Y}$. Let $\hat{P}_r := U_r\Sigma_r^{1/2} = \begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \end{pmatrix}$, so that $\bar{Y}_r = \hat{P}_r\hat{P}_r^T$. Let $\hat{Q} := \operatorname{argmin}_{Q^TQ=I}\|\hat{P}_2Q - A\|_F^2$. Then $\hat{Q} = V_QU_Q^T$, where $A^T\hat{P}_2 = U_Q\Sigma_QV_Q^T$. Finally, we let $X = \hat{P}_1\hat{Q}$ be our estimate for the sensors.