Abstract
Differential algebra provides an algebraic viewpoint on nonlinear differential systems. The motivating questions for this talk are:

- How do we define the general solution of a nonlinear equations
- What are the conditions for a differential system to have a solution
- How do we measure the degrees of freedom for the solution set of a differential system

Theory and algorithms for those are extensions of commutative algebra (prime ideal decomposition, Hilbert polynomials) and Groebner bases techniques.

The library diffalg in Maple supports this introduction to constructive differential algebra. It has been developed by F. Boulier (1996) and the speaker afterwards. A recent extension of differential algebra to non-commutative derivations, and its implementation in diffalg, allow to treat systems bearing on differential invariants.

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1 Sample problems

Envelope

Consider a family of curves
\[ y = cx + c^2 \]

What is their envelope?

The differential equation satisfied by this family is:
\[ \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = y \]

General solution: \( y = cx + c^2 \)

Singular solution: \( y = -1/4x^2 \)

\[ \text{Kepler } \Rightarrow \text{ Newton} \]

Kepler’s observational laws:
\[ K_1 \] the planets move along ellipses with the sun as focus
\[ r = p + e x \]
where \( r^2 = x^2 + y^2 \)
and \( \dot{p}, \dot{e} = 0 \)
\[ K_2 \] the vector from the sun to the planet sweeps equal area in equal times
\[ x \dot{y} - \dot{x} y = s \]
with \( \dot{s} = 0 \)

Newton’s gravitational laws:
\[ N_1 \] the acceleration is inversely proportional to the square of the distance to the sun.
\[ \frac{d}{dt} (r^2 a) = 0 \]
where \( a^2 = (\dot{x}^2 + \dot{y}^2) \)
\[ N_2 \] the acceleration vector is directed towards the sun
\[ x \ddot{y} - \dot{x} \dot{y} = 0 \]

\[ K_2 \Rightarrow N_2, \quad K_1, K_2 \Rightarrow N_1 ? \]

Membership \[ [\text{Wu 91}] \]

Orthogonal waves \[ [\text{G. Metivier}] \]

\[ s (\phi_{xx} + \phi_{yy}) + s_x \phi_x + s_y \phi_y + \phi = 0 \]
\[ s (\psi_{xx} + \psi_{yy}) + s_x \psi_x + s_y \psi_y + \psi = 0 \]
\[ \psi_x \phi_x + \psi_y \phi_y = 0 \]

- Is there a solution? \hspace{1cm} consistency
- What are the degrees of freedom \hspace{1cm} completion
\[ s = f_1(y) + f_2(y) x + s_{20} \frac{x^2}{2} + s_{30} \frac{x^3}{3} \ldots \]
\[ \psi = f_3(y) + c_1 x + \psi_{20} \frac{x^2}{2} + \psi_{11} x y + \psi_{20} \frac{x^2}{2} + \phi_{11} x y + \ldots \]
\[ \phi = f_4(y) + \phi_{10} x + \phi_{01} y + \phi_{20} \frac{x^2}{2} + \phi_{11} x y + \ldots \]

4 functions of 1 variable, 1 constant.

- Conditions on \( s \)

**Equivalence**

\[ y'' = f(x, y, y') \quad \exists? \xi, \eta \quad Y = \xi(x, y) \]

\[ y_2 = f(x, y_0, y_1) \quad \exists? \xi, \eta \quad Y_2 = 0 \]

\[ \begin{align*}
  \xi y_1 &= 0, & \eta y_1 &= 0,
  
  (\eta y_0, \xi x - \xi y_0, \eta x) f + (\eta y_0, \xi y_0 - \eta y_0, \xi y_0 y_0) y_1^3 
  
  + (2 \eta y_0, \xi y_0 - 2 \eta y_0, \xi y_0 y_0 + \eta y_0, \xi x) y_1^2 
  
  + (2 \eta y_0, \xi x - 2 \eta y_0, \xi y_0 y_0 + \eta y_0, \xi x y_0) y_1 - \eta y_0, \xi x x + \eta y_0 \xi x &= 0
\end{align*} \]

Differential indeterminates: \( \xi, \eta, f \), functions of \( x, y_0, y_1 \)

Derivations: \( \partial_x = \frac{\partial}{\partial x} \), \( \partial_{y_0} = \frac{\partial}{\partial y_0} \), \( \partial_{y_1} = \frac{\partial}{\partial y_1} \)

Differential indeterminates: \( \xi, \eta, f \)

Consider \( \partial_{y_0}, \partial_{y_1} \), and \( D_x = \partial_x + y_1 \partial_{y_0} + f \partial_{y_1} \)

\[ 
\begin{align*}
  \partial_{y_0} D_x - D_x \partial_{y_0} &= f_{y_0} \partial_{y_0}, & \partial_{y_1} D_x - D_x \partial_{y_1} &= \partial_{y_0} + f_{y_1} \partial_{y_1}
\end{align*} \]

\[ \begin{align*}
  \xi y_1 &= 0, & \eta y_1 &= 0,
  
  \xi y_0 y_0 - \eta y_0 \xi x &= 0
\end{align*} \]

General case:

\[ f_{y_1 y_1 y_1 y_1} = 0, f_{x y y_1 y_1} = 4 f_{x y_0 y_1} + f_{y_1} f_{x y_1 y_1} - 4 f_{y_1} f_{x y_0 y_1} - 6 f_{y_0 y_0} + 3 f_{y_0} f_{y_0}
\]

Fiber preserving case (\( \xi y_0 = 0 \))

\[ \begin{align*}
  f_{y_1 y_1 y_1} &= 0, & f_{x y y_1} &= f_{y_0 y_1}, & f_{x y_0 y_1} &= 2 f_{y_0 y_0} - f_{y_1 y_1} f_{y_0} + f_{y_1} f_{y_0 y_1}
\end{align*} \]
2 Software

Software for nonlinear differential systems

\textit{diffalg}: Differential Algebra

1996 Rosenfeld-Gröbner algorithm [BLOP 1997]
by F. Boulier (1996) then at SCG, U. of Waterloo.

1998 Singular solutions [H 99]; Efficiency improvement [H 00]
by E. Hubert then at SCG, UWaterloo.

1999 Redesign of help pages.

2004 Non-commuting derivations [H 2005]
\url{http://www.inria.fr/cafe/Evelyne.Hubert/diffalg}

\textbf{BLAD}: Bibliothèques Lilloises d’Algèbre Différentielle

C libraries distributed under Gnu Lesser General Public License
\url{http://www2.lifl.fr/~boulier/BLAD} by F. Boulier

\textbf{RegularChains} for polynomial systems

by F. Lemaire & M. Moreno Maza \textit{et al.} U.WO

\textbf{RIF}: Reduced Involutive Forms

A. Wittkopf, G. Reid

\textbf{CRACK}: PDE solver

T. Wolf (\url{http://lie.math.brocku.ca/crack})

Software for linear functional systems

\texttt{kan/sm1} by N. Takayama (\url{http://www.math.kobe-u.ac.jp/KAN})

D-Macaulay by A. Leykin & H. Tsai (\url{http://www.ima.umn.edu/~leykin/Dmodules})

Cocoa, \url{http://cocoa.dima.unige.it/}

Plural:Singular by V. Levandovskyy & H. Schönemann (\url{http://www.singular.uni-kl.de/plural})

In Maple: Groebner by F. Chyzak; OreModule with A. Quadrat & D. Robertz.
(\url{http://wwwb.math.rwth-aachen.de/OreModules})
3 Ring of differential polynomials

Classical construction [Ritt 51, Kolchin 73]

\[ F = \mathbb{Q} \text{ or } \mathbb{Q}(x, y) \]
\[ \Delta = \{ \delta_1, \ldots, \delta_m \} \] derivations on \( F \)
\[ \delta_i (a + b) = \delta_i(a) + \delta_i(b) \]
\[ \delta_i(ab) = a\delta_i(b) + \delta_i(a)b \]

\[ s \left( \phi_{xx} + \phi_{yy} \right) + s \phi_x + s y \phi_y + \phi = 0 \]
\[ s \left( \psi_{xx} + \psi_{yy} \right) + s x \psi_x + s y \psi_y + \psi = 0 \]
\[ \psi_x \phi_x + \psi_y \phi_y = 0 \]

\[ Y = \{ s, \phi, \psi \} \]
\[ \tilde{Y} = \{ y_1, \ldots, y_n \} \]

\[ F[s, s_x, s_y, s_{xx} \ldots] = F[s, \phi, \psi] \]
\[ s_{xx}y \sim s_{x}y_{x} \sim s_{(2,1)} \]

\[ \frac{\partial}{\partial x} (s_{xy}) = s_{xxy} \sim s_{(1,1)} \]
\[ \delta_{1}(s_{(1,1)}) = \delta_{(2,1)} \]

\[ \frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial y} \frac{\partial}{\partial x} \ delta \delta \]

\[ \delta_{i} \delta_{j} = \delta_{j} \delta_{i} \]

Declare it in Maple

```
> with(diffalg);
[RosenfeldGroebner, characters, delta, polynomial, denote, derivatives, differential ring, differentiate, equations, essential components, field extension, greater, inequations, initial, initial conditions, is prime, leader, power series solution, preparation polynomial, print ranking, rank, reduce, reduced form, rewrite rules, separate]
> R := differential_ring( ranking=[[s,phi,psi]], derivations=[x,y], notation=jet );
R := differential polynomial ring

> differentiate( s[x,y], x, R );
s_{x.x.y}

> denote( s[x,y], vjet, R );
s_{1,1}

> denote( s[x,y], diff, R );
\frac{\partial^2}{\partial x \partial y} (x, y)
```

Rankings \( \prec \) on \( F[\tilde{Y}] \)

A total order on \( \tilde{Y} \) = \( \{ y_{\alpha} \mid \alpha \in \mathbb{N}^{m}, y \in \tilde{Y} \} \) s.t.

\[ y_{\alpha} \prec y_{\alpha + \gamma} \]
\[ y_{\alpha} \prec z_{\beta} \Rightarrow y_{\alpha + \gamma} < z_{\beta + \gamma} \]

| \( \alpha \) | = \( \alpha_{1} + \ldots + \alpha_{m} \)

Orderly ranking: \( |\alpha| < |\beta| \Rightarrow y_{\alpha} < z_{\beta}, \forall y, z \)

Semi-orderly ranking: \( |\alpha| < |\beta| \Rightarrow y_{\alpha} < y_{\beta} \)

Elimination ranking: \( y_{\alpha} \prec z_{\beta}, \forall \alpha, \beta \)
Rankings with \textit{diffalg}

\[
DY := \{seq(seq(seq(u[x[i,y[j-i]],i=0..j],j=0..2),u=[s,phi,psi]),i=0..2),u=[s,phi,psi])
\]

\[
R2 := \text{differential\_ring(ranking=[grlex[s],grlex[phi,psi]],derivations=[x,y])};
\]

\[
\text{sort}(DY, (a,b) \rightarrow \text{greater}(b,a,R2));
\]

\[
R1 := \text{differential\_ring(ranking=[s,phi,psi],derivations=[x,y])};
\]

\[
\text{sort}(DY, (a,b) \rightarrow \text{greater}(b,a,R1));
\]

Non-commuting derivations

\[
y_2 = f(x, y_0, y_1) \quad X = \xi(x, y_0) \quad Y_2 = 0
\]

\[
Y_0 = \eta(x, y_0)
\]

\[
\begin{cases}
\xi_{y_1} = 0, & \eta_{y_1} = 0, \\
\xi_x \eta_{xx} - \xi_{xx} \eta_x = 0
\end{cases}
\]

Differential indeterminates : \(\xi, \eta, f\)

Derivations \(\partial_{y_0}, \partial_{y_1}\), and \(D_x\)

\[
\partial_{y_0} D_x - D_x \partial_{y_0} = f_{y_0} \partial_{y_1},
\]

\[
\partial_{y_1} D_x - D_x \partial_{y_1} = \partial_{y_0} + f_{y_1} \partial_{y_1}
\]

\[
\partial_{y_1} \partial_{y_0} - \partial_{y_0} \partial_{y_1} = 0
\]

Differential polynomial ring \(K[Y]\) with non commuting derivations \([H05]\)

\[
Y = \{y_1, \ldots, y_n\}
\]

\[
D = \{\delta_1, \ldots, \delta_m\}
\]

\[
K[y_\alpha \mid \alpha \in \mathbb{N}_m, \ y \in Y]
\]

Example (\(m=2\)):

\[
\delta_1 y_{(1,1)} = y_{(2,1)}
\]

\[
\delta_2 y_{(1,1)} = \delta_2 \delta_1 y_{(0,1)} = \delta_1 \delta_2 y_{(0,1)} + c_{121} \delta_1 y_{(0,1)} + c_{122} \delta_2 y_{(0,1)}
\]

\[
\delta_1 y_{(2,2)} = \delta_1 y_{(0,1)} + c_{121} \delta_1 y_{(1,1)} + c_{122} \delta_2 y_{(0,2)}
\]

\[
y_{(1,2)} = \delta_1 y_{(1,1)} + c_{121} y_{(1,1)} + c_{122} y_{(0,2)}
\]

If the \(c_{ijl}\) satisfy

\[
\delta_i y_{(\alpha)} = c_{i\alpha} y_{(\beta)} + c_{i\beta} y_{(\gamma)}
\]

\[
\sum_{\alpha=1}^m c_{i\alpha} y_{(\alpha)} + c_{i\beta} y_{(\beta)} = -c_{ijl} y_{(\alpha)} - c_{ijkl} y_{(\beta)} - c_{ijkl} y_{(\gamma)}
\]

\[
\sum_{l=1}^m c_{ijkl} y_{(\alpha)} < y_{(\alpha) + \epsilon_i + \epsilon_j}
\]

\[
|\alpha| < |\beta| \Rightarrow y_{\alpha} < y_{\beta},
\]

\[
y_{\alpha} < z_{\beta} \Rightarrow y_{\alpha + \gamma} < z_{\beta + \gamma},
\]

\[
\sum_{l=1}^m c_{ijkl} y_{(\alpha)} < y_{\alpha + \epsilon_i + \epsilon_j}
\]
then $\delta_i \delta_j(p) = \delta_j \delta_i(p) + \sum_{l=1}^{m} c_{ijl} \delta_l(p) \quad \forall p \in \mathbb{K}[y_\alpha \mid \alpha \in \mathbb{N}^m] = \mathbb{K}[\mathcal{Y}]$

With $\text{diffalg}$

$$\partial_{y_0} D_x - D_x \partial_{y_0} = f_{y_0}\partial_{y_1}, \quad \partial_{y_1} D_x - D_x \partial_{y_1} = \partial_{y_0} + f_{y_1}\partial_{y_1}$$

$$> R := \text{differential_ring}(\text{ranking}=[x_1,y_\alpha,f], \text{derivations}=[x,y_0,y_1], \text{commutations}=[y_0,x]=[0,0,f[y_0]], [y_1,x]=[0,1,f[y_1]]) :$$

$$> \text{differentiate( differentiate( eta, y_0, R ), x, R );}$$

$$> \text{differentiate( eta, x, y_0, R );}$$

$$> \text{differentiate( eta, y_0, x, R );}$$

$\eta_{x,y_0} + f_{y_0} \eta_{y_1}$

Leader, initial, separant

- $\mathcal{Y} = \{y_1, \ldots, y_n\}$,
- $\mathcal{D} = \{\delta_1, \ldots, \delta_m\}$,
- $\mathbb{F}[\mathcal{Y}]$,
- $\prec$

- $\text{Nota}^{\phi}$: $\delta^\alpha = \delta_1^{\alpha_1} \ldots \delta_m^{\alpha_m}$

$$\delta^{\alpha+\beta} \neq \delta^{\alpha+\beta} \text{ but } \delta^{\alpha+\beta} = \delta_1^{\alpha_1+\beta_1} + \ldots + \delta_m^{\alpha_m+\beta_m}$$

- $p \in \mathbb{K}[\mathcal{Y}] \setminus \mathbb{K}$

$$p = i_p y_\alpha^d + c_1 y_\alpha^{d-1} + \ldots + c_d \quad c_i \prec y_\alpha$$

$$y_\alpha \text{ leader lead}(p) \quad i_p \text{ initial init}(p) \quad s_p \text{ separant sep}(p)$$

- $s_p = \frac{\partial p}{\partial y_\alpha} = d i_p y_\alpha^{d-1} + (d-1) c_1 y_\alpha^{d-2} + \ldots + c_d$,

Prop:

$$\text{lead}(p) = y_\alpha \quad \Rightarrow \quad \delta^{\beta}(p) = \text{sep}(p) y_\alpha + \ldots \prec y_\alpha + \beta$$
4 Differential ideals

Differential Ideals

\[ \mathcal{Y} = \{ y_1, \ldots, y_n \}, \quad \mathcal{D} = \{ \delta_1, \ldots, \delta_m \}, \quad \mathbb{F}[\mathcal{Y}] \]

\[ \{ p_1, \ldots, p_k \} \subset \mathbb{F}[\mathcal{Y}] \]

\( I, \) a differential ideal of \( \mathbb{F}[\mathcal{Y}] \):

\( \bullet \ a \in I \Rightarrow \delta a \in I, \forall \delta \in \Delta \)
\( \bullet \ I \) is an ideal

\( J, \) radical differential ideal of \( \mathbb{F}[\mathcal{Y}] \):

\( \bullet \ a^k \in J \Rightarrow a \in J \)
\( \bullet \ J \) is a differential ideal

\( P, \) prime differential ideal of \( \mathbb{F}[\mathcal{Y}] \):

\( \bullet \ ab \in P \Rightarrow a \in P \) or \( b \in P \)
\( \bullet \ P \) is a differential ideal

Differential Nullstellensatz

\[ p_1, \ldots, p_r \in \mathbb{F}[\mathcal{Y}] \]

Note:

\(-\) if \( q \in \llbracket p_1, \ldots, p_r \rrbracket \) then \( q^e = \sum c_i \alpha \delta^\alpha p_i \) and \( q \) vanishes on all the common zeros of \( p_1, \ldots, p_r \).

\(-\) if \( 1 \in \llbracket p_1, \ldots, p_r \rrbracket \) then \( 1 = \sum c_i \alpha \delta^\alpha p_i \) so that \( p_1, \ldots, p_r \) have no common zero.

Theo:

\(- p_1, \ldots, p_k \) admit a common zero iff \( 1 \notin \llbracket p_1, \ldots, p_r \rrbracket \)
\(- q \in \mathbb{F}[\mathcal{Y}] \) vanishes on all the common zeros of \( p_1, \ldots, p_k \) iff \( q \in \llbracket p_1, \ldots, p_k \rrbracket \).

Ritt-Raudenbush Theorem

Theo: A radical differential ideal \( J \) of \( \mathbb{F}[\mathcal{Y}] \) is

\(-\) finitely generated: \( J = \llbracket p_1, \ldots, p_r \rrbracket \)
for some \( p_1, \ldots, p_r \in \mathbb{F}[\mathcal{Y}] \).

Differential ideals need not be finitely generated.

\(-\) the intersection of finitely many prime differential ideals.

The irredundant decomposition \( J = \bigcap_{i=1}^r P_i \) is unique.

Ex: \( \llbracket y_x^2 + x y_x - y \rrbracket = \llbracket y_x^2 + x y_x - y, y_{xx} \rrbracket \cap \llbracket 4y + x^2 \rrbracket \).
Saturation ideals

- \( h \in \mathbb{F}[[Y]] \quad I : h^\infty = \{ q \mid \exists k \text{ s.t. } h^k q \in I \} \)

- \( H = \{ h_1, \ldots, h_s \} \quad I : H^\infty = I : h^\infty \text{ where } h = h_1 \ldots h_s \)

NOTE: The zeros of \([p] : h^\infty\) are the zeros of \(p\) that don’t vanish on \(h\), except for some adherent piece.

EX: \([y_x^2 + x y_x - y] = [y_x^2 + x y_x - y] : (2y_x + x)^\infty \cap [4y + x^2]\).

## 5 Representation of radical differential ideals

**Purpose**

Given \(p_1, \ldots, p_r\) in \(\mathbb{F}[[Y]]\) we want to compute a representation of \([p_1, \ldots, p_r]\) that allows to

- test membership to \([p_1, \ldots, p_r]\)

- *measure* the zero set \hspace{1cm} (completion)

- compute \([p_1, \ldots, p_r] \cap \mathbb{F}[[Z]]\) \hspace{1cm} (elimination)

In a factorisation free way.

There is no strict analogue of Gröbner bases.

Differential ideals do not admit in general

- a finite set

- that is generating for the differential ideal

- and reduces to zero the elements of the differential ideal

Characterisable differential ideals are defined by

- a finite set

- that reduces to zero the elements of the differential ideal

- generate the ideal *outside of some hypersurface*

Prime differential ideals are characterizable

Radical diff. ideal are \(\cap^o\) of characterisable diff. ideals
Characteristic Decomposition

ALGO: (Rosenfeld-Groebner in diffalg)

\[
\begin{align*}
\text{In:} & \quad \{p_1, \ldots, p_k\} \subset F[Y], < \\
\text{Out:} & \quad C_1, \ldots, C_r \text{ s.t.} \quad [p_1, \ldots, p_k] = [C_1]:S_1^\infty \cap \ldots \cap [C_r]:S_r^\infty
\end{align*}
\]

Membership: \( p \in [C_i]:S_i \iff p \rightarrow_c 0 \)

Completion: If \( \prec \) orderly then \( [C_i]:S_i^\infty \cap F[D_{<y_i}] = (D_{<y_i}C_i):S_i^\infty \)

Elimination: If \( \prec \) eliminates \( Y \setminus Z \) and \( C'_i = C_i \cap F[\bar{Z}] \) then \( [C_i]:S_i^\infty \cap F[\bar{Z}] = [C'_i]:S_i^\infty \)

\textbf{Output characteristic sets} \( C \)

- \( C \) is coherent:
  \[
  a, b \in C, \quad \text{lead}(a) = y_\alpha, \text{lead}(b) = y_\beta \\
  \gamma = \alpha + \bar{\alpha} = \beta + \bar{\beta} \quad \Rightarrow \quad s_a \delta^\beta b - s_b \delta^\alpha a \in (D_{<y_i}C_i):S^\infty
  \]
  
  Recall:
  \[
  \begin{align*}
  \delta^\beta b &= s_b y_\gamma + \ldots \\
  \delta^\alpha a &= s_a y_\gamma + \ldots
  \end{align*}
  \]
  
  A generalisation of
  \[
  \begin{align*}
  u_x &= f(x, y) \\
  u_y &= g(x, y)
  \end{align*}
  \Rightarrow \quad f_y - g_x = 0
  \]

- \( C \) is a differential triangular set:
  \[
  a, b \in C, \quad \text{lead}(a) = y_\alpha, \text{lead}(b) = y_\beta \quad \Rightarrow \quad \beta \neq \alpha + \gamma
  \]
  
  - \([C]:S^\infty\) is a radical differential ideal.

- \( p \rightarrow_c q \) means:
  - \( q \) is free of proper derivatives of \( \text{lead}(c) \)
    \[
    s_c^e p = q_1 \quad \text{mod} \ (\delta^\beta c, \ldots, \delta^\gamma c)
    \]
  - the degree of \( q \) in \( \text{lead}(p) \) is lower than that of \( p \)
    \[
    i_p^f q_1 = q \quad \text{mod} \ (p)
    \]
Envelope

> R := differential_ring(ranking=[c,y], derivations=[x]);
> C := Rosenfeld_Groebner([y - cx - c^2, cx], R);

\[
C := \text{characterisable}
\]

> equations(G);

\[
\begin{bmatrix}
- y x + c, y^2 + x y - y
\end{bmatrix}
\]

> p := equations(G[1])[-1];

\[
p := y^2 + x y - y
\]

> C := Rosenfeld_Groebner([p], R);

\[
C := \begin{bmatrix}
\text{characterisable, characterisable}
\end{bmatrix}
\]

> equations(C), inequations(C)

\[
\begin{bmatrix}
y^2 + x y - y, [4y + x^2], [2y + x]
\end{bmatrix}
\]

That is:

\[
[p] = \begin{bmatrix}
(2y + x)^\infty \cap [4y + x^2]
\end{bmatrix}
\]

Kepler ⇒ Newton

> R := differential_ring(ranking=[[x,y], [a,r], [p,e,s]], derivations=[t], parameters=[p,e,s]);
> K := \{ a^2 - x^2 t, t - y^2 t, r^2 - x^2 - y^2, r - p + e x, x y - y x t - s \};
> C := Rosenfeld_Groebner(K, \{ p, e, s \}, R);

\[
C := \text{characterisable}
\]

> N := differentiate(r^2 a, t, R);

\[
N := r (a_t r + 2 r t a)
\]

> reduce(N, C[1]);

\[
0
\]

With

\[
\begin{align*}
[K] : (pes)^\infty &= [c x + r - p, e s y - p r r_t, r^2 p^2 r_t^2 - s^2 (c^2 r^2 + r^2 - 2 r p + p^2), r^4 p^2 a^2 - s^4] : (epsarr_t)^\infty
\end{align*}
\]

Orthogonal Waves

\[
\begin{align*}
s (\phi_{xx} + \phi_{yy}) + s_x \phi_x + s_y \phi_y + \phi &= 0 \\
s (\psi_{xx} + \psi_{yy}) + s_x \psi_x + s_y \psi_y + \psi &= 0 \\
\psi_x \phi_x + \psi_y \phi_y &= 0 \\
s_x, s_y, \phi_x, \phi_y, \psi_x, \psi_y &\neq 0
\end{align*}
\]

> with(diffalg)
> R := differential_ring(ranking=[[s,phi,psi], derivations=[x,y]]):
> S := [s*phi[x,x]+phi[y,y]+s[x]*phi[x]+...];
> H := [s[x], s[y], phi[x],...];
> C := Rosenfeld_Groebner(S, H, R);

\[
C := \text{characterisable}
\]
\[ C = \{ s_{xx} = \ldots, \psi_{xx} = \ldots, \psi_{xy} = \ldots, \phi_x = \ldots \} \]

\[
\begin{align*}
\psi &= \psi_{00} + \psi_{10} x + \psi_{01} y + \psi_{20} \frac{x^2}{2} + \psi_{11} x y + \psi_{02} \frac{y^2}{2} + \cdots \\
\phi &= \phi_{00} + \phi_{10} x + \phi_{01} y + \phi_{20} \frac{x^2}{2} + \phi_{11} x y + \phi_{02} \frac{y^2}{2} + \cdots \\
\end{align*}
\]

\[
\begin{align*}
s &= f_1(y) + f_2(y) x + s_{20} \frac{x^2}{2} + s_{30} \frac{x^3}{6} + \cdots \\
\psi &= f_3(y) + c_1 x + \psi_{20} \frac{x^2}{2} + \psi_{11} x y + \psi_{20} \frac{y^2}{2} + \cdots \\
\phi &= f_4(y) + \phi_{10} x + \phi_{01} y + \phi_{20} \frac{x^2}{2} + \phi_{11} x y + \cdots \\
\end{align*}
\]

Challenge Computational Problem

\[
\begin{align*}
\psi_x \phi_x + \psi_y \phi_y &= 0 \\
\end{align*}
\]

6 Symmetric Systems

\[
\begin{align*}
S &= \left\{ 
\begin{align*}
(s (\phi_{xx} + \phi_{yy}) + s_x \phi_x + s_y \phi_y + \phi) &= 0 \\
(s (\psi_{xx} + \psi_{yy}) + s_x \psi_x + s_y \psi_y + \psi) &= 0 \\
\psi_x \phi_x + \psi_y \phi_y &= 0
\end{align*}
\right. \\
\end{align*}
\]

Idea: factor out the symmetry [Mansfield 01], [Lisle & Reid 06] [15pt]
Tool: moving frame construction [Fels & Olver 99]

\[
\begin{align*}
(x, y, s, \phi, \psi) \quad \xrightarrow{g_{\alpha, \beta, \rho, \tau, \mu, \nu, a, b}} \quad (X, Y, S, \Phi, \Psi) \quad \beta^2 + \alpha^2 = 1
\end{align*}
\]

\[
\begin{align*}
X &= \frac{\alpha}{\rho} x - \frac{\beta}{\rho} y + \frac{\alpha}{\rho} \\
Y &= \frac{\alpha}{\rho} x + \frac{\beta}{\rho} y + \frac{b}{\rho} \\
S &= \frac{s}{\rho^2} \\
\Phi &= \frac{\phi}{\mu} \\
\psi &= \frac{\psi}{\nu}
\end{align*}
\]

\[
\begin{align*}
S_X &= \frac{\beta}{\rho} s_x - \frac{\alpha}{\rho} s_y \\
S_Y &= \frac{\alpha}{\rho} s_x + \frac{\beta}{\rho} s_y \\
\Phi_X &= \frac{\alpha}{\mu} \phi_x - \frac{\beta}{\mu} \phi_y \\
\Phi_Y &= \frac{\alpha}{\mu} \phi_x + \frac{\beta}{\mu} \phi_y
\end{align*}
\]

\[
\begin{align*}
\cdots
\end{align*}
\]
All differential invariants can be written in terms of the fundamental invariants [Vessiot, Groebner]

\[ s_1 := \frac{s_x^2 + s_y^2}{4 s}, \]
\[ s_2 := s_{xy}(s_y^2 - s_x^2) + s_x s_y (s_{xx} - s_{yy}), \]
\[ s_3 := \frac{s_{xy}^2 s_{yy} + s_y^2 s_{xx} - 2 s_x s_y s_{xy}}{8 s^3} - s_1, \]
\[ \psi_1 := \frac{s_y \psi_x - s_x \psi_y}{2 s_1 \psi}, \]
\[ \phi_1 := \frac{s_y \phi_x - s_x \phi_y}{2 s_1 \phi}, \]
\[ \psi_2 := \frac{s_y \psi_x + s_x \psi_y}{2 s_1 \psi}, \]
\[ \phi_2 := \frac{s_x \phi_x + s_y \phi_y}{2 s_1 \phi}. \]

(s_1, s_2, s_3 depend only on s and derivatives)

and their derivatives with respect to the invariant derivations:

\[ \left( \begin{array}{c} \delta_1 \\ \delta_2 \end{array} \right) = \pm \sqrt{s \left( \frac{s_x^2 + s_y^2}{s_x^2 - s_y^2} \right)} \left( \begin{array}{c} -s_y \\ s_x \end{array} \right) \left( \frac{\partial}{\partial y} \right) \]

\[ \mathcal{P} = (x, y, s - 1, \phi - 1, \psi - 1, s_x) \]

Invariants: \( s_1 = \iota(s_y)/2, s_2 = \iota(s_x^2), s_3 = \iota(s_x s_y^2), \quad s_1 = \iota(s_x x), \quad s_2 = \iota(s_x y), \quad s_3 = \iota(s_x y^2) \) and \( \phi_1 = \iota(\phi_x), \quad \psi_1 = \iota(\psi_x) \).

We can write \( \mathcal{S} \) in terms of \( \mathcal{Y} = \{ s_1, s_2, s_3, \phi_1, \phi_2, \psi_1, \psi_2 \} \) and \( \Delta = \{ \delta_1, \delta_2 \} \) expressions of \( s_1, s_2, s_3, \phi_1, \phi_2, \psi_1, \psi_2 \) not needed

\[ \mathcal{S} \left\{ \begin{array}{l}
\delta_1(\phi_1) + \delta_2(\phi_2) + \delta_1^2(\phi_2) - s_2 \phi_1 + (2 s_1 + s_3) \phi_2 + 1 = 0,
\delta_1(\psi_1) + \delta_2(\psi_2) + \psi_1^2 + \psi_2^2 - s_2 \psi_1 + (2 s_1 + s_3) \psi_2 + 1 = 0,
\phi_1 \psi_1 + \phi_2 \psi_2 = 0.
\end{array} \right. \]

but now

\[ \delta_1 \delta_2 - \delta_2 \delta_1 = s_3 \delta_1 + s_2 \delta_2 \]

and the fundamental invariants are not differentially independent.

The syzygies are

\[ \mathcal{Z} \left\{ \begin{array}{ll}
\delta_1(s_1) &= s_1 s_2 \\
\delta_1(s_2) - \delta_2(s_3) &= s_3^2 + s_2^2 + s_1 (s_2 + s_3) \\
\delta_1(\phi_1) - \delta_2(\phi_2) &= \phi_1 s_3 + \phi_2 s_2, \\
\delta_1(\psi_1) - \delta_2(\psi_2) &= \psi_1 s_3 + \psi_2 s_2. 
\end{array} \right. \]

Project status

Rational Invariants of a Group action. Construction and Rewriting, with I. Kogan, JSC, in press.

Smooth and Algebraic Invariants. Local and Global construction, with I. Kogan, submitted.


Software: aida, diffalg
inria.fr/cafe/Evelyne.Hubert/aida
inria.fr/cafe/Evelyne.Hubert/diffalg

Thanks.