# What should a software package for Numerical Algebraic Geometry be?


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# Outline

- Characteristics of good (numerical) software
  - Reliable, Accurate, Fast, Modular, User Friendly
  - What does it mean to “solve” equations
    - Double precision, multiprecision, adaptive multiprecision
- Functionality of a NumAlgGeom package
  - Base level operations
    - Numeric & symbolic
  - Mezzo level operations
  - Algorithms for solving polynomial systems
    - Zero dimensional vs. positive dimensional
    - Reduced and non-reduced solutions
  - Higher level algorithms
    - Intersections, extraction of reals, computing genera
- Interfaces
  - How should this be packaged for widest/best usability?

# Characteristics of good software

- Reliable
  - with clear failure signals
- As fast as possible
  - with time to completion estimates on big jobs
- Accurate
  - With error estimates
- Modular
- User friendly

# Trade-offs

- Reliability and accuracy cost speed
- Numerical accuracy depends on precision
  - High degree systems or bad scaling may require *multiprecision*
    - Expensive, but available
    - Affordable if used judiciously
- Two approaches
  - Do the best you can in fixed precision
    - If double precision fails, try more digits
  - Adaptive Multiprecision
    - Adjust precision as necessary using computable bounds on the numerical behavior of the system

# What does "solve" mean?

- Given $p(z) \in \mathbf{C}[z]$, "solve" $p(z)=0$.
- "solve" #1
  - Specify $e \in \mathbf{R}^+$
  - Find $z \in S$, $S=\{s \in \mathbf{C} : |p(s)|=e\}$
    - One point in each connected component?
  - Note: $p(z)=0$ and $f(z)=2p(z)=0$ may have different solutions
- "solve" #2
  - Specify $e \in \mathbf{R}^+$
  - Find $z \in \mathbf{C}$ such that $|z-s|=e$ where $p(s)=0$
    - One z near each distinct solution s
    - *m* z's near each solution s of multiplicity *m*
  - Note: $p(z)=0$ and $f(z)=p(2z)=0$ may have different solutions
  - Weak version: do your best at the above and report an estimate of $|z-s|$.

# Ex 1: Monic Chebyshev Polynomials

$$T_0 = 2,\ T_1 = z$$

$$T_k = zT_{k-1} - \tfrac{1}{4}T_{k-2}$$

$$T_{10} = z^{10} - \tfrac{5}{2}z^8 + \tfrac{35}{16}z^6 - \tfrac{25}{32}z^4 + \tfrac{25}{256}z^2 - \tfrac{1}{512}$$

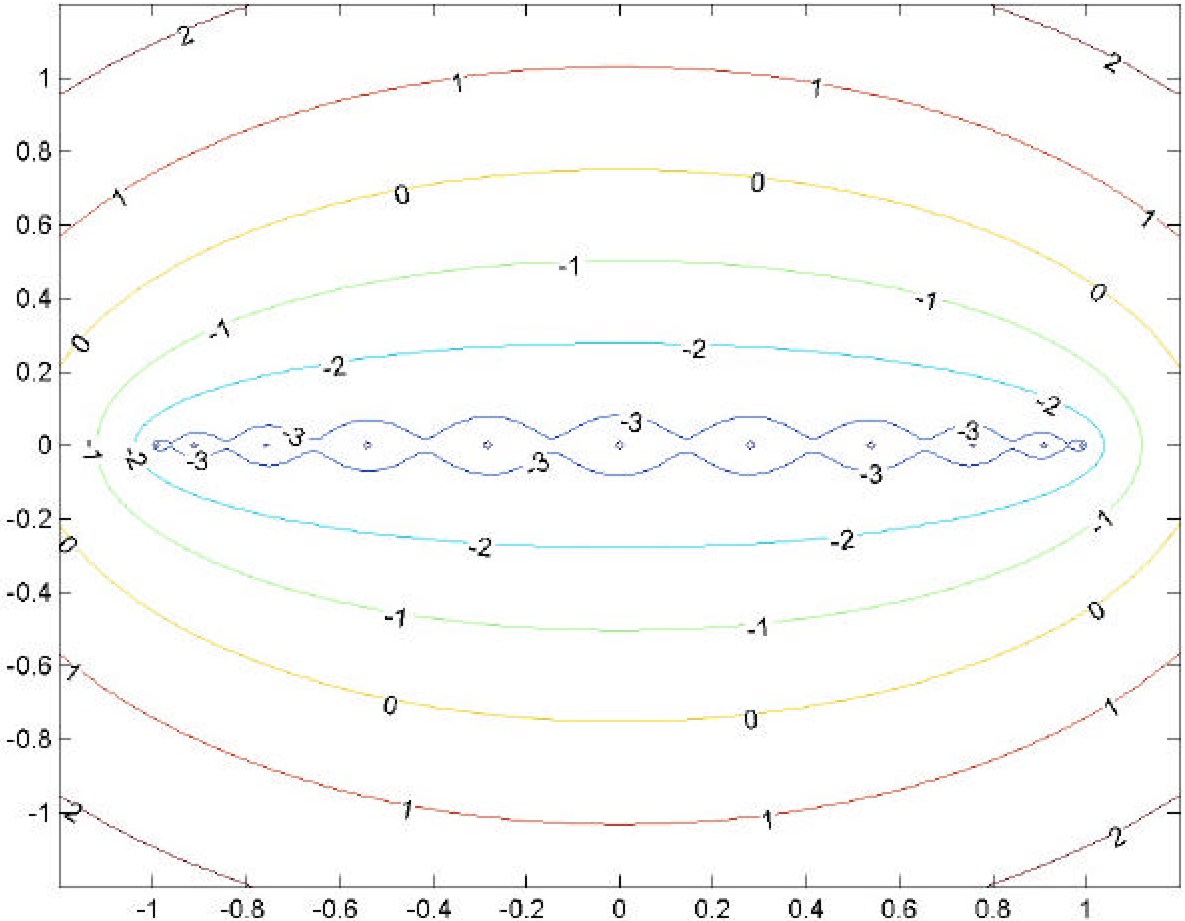
$$T_{11} = z^{11} - \tfrac{11}{4}z^9 + \tfrac{11}{4}z^7 - \tfrac{77}{64}z^5 + \tfrac{55}{256}z^3 - \tfrac{11}{1024}z$$

Smallest coefficient is $2^{-k+1}$ or $k2^{-k+1}$

For $z \in [-1,1]$, $|T_k(z)| \le 2^{-k+1}$,

and reaches this limit $k+1$ times

# $\log_{10}|T_{11}(z)|$ contour plot




- At e=.001, 1 solution zone.
- At e=.0001, 11 solution zones.

→ Precision matters.

# Wilkinson polynomials

$$W_n = (z-1)(z-2)\cdots(z-n)$$

$$W_{11}(z) = z^{11} - 66z^{10} + 1925z^9 - 32670z^8 + 357423z^7$$
$$-2637558z^6 + 13339535z^5 - 45995730z^4$$
$$+105258076z^3 - 150917976z^2 + 120543840z - 39916800$$

```
>> disp( double(w11) - single(w11) )
   0   0   0   0   0   0   0   -2   -4   8   0   0
```

# Symbolic results

- Solving $W_{11}(z)=0$
  - maple('solve(expand((z-1)*(z-2)*(z-3)*
    (z-4)*(z-5)*(z-6)*(z-7)*(z-8)*(z-9)*(z-10)*(z-11)))')
  - 1,2,3,4,5,6,7,8,9,10,11
  - Beautiful!
- Solving $W_{11}(z)+1=0$
  - maple('solve(expand((z-1)*(z-2)*(z-3)*
    (z-4)*(z-5)*(z-6)*(z-7)*(z-8)*(z-9)*(z-10)*(z-11)+1))')
  - RootOf(_Z^11-66*_Z^10+...-39916799,index = 1),
  ...RootOf(...,index = 11)
  - Not so useful...
- Numerically, these are of the same difficulty

# Numeric roots of $W_{11}(z)=0$ (Matlab)

- Plot of Matlab results
  - >> roots(single(w11)) and >> roots(double(w11))

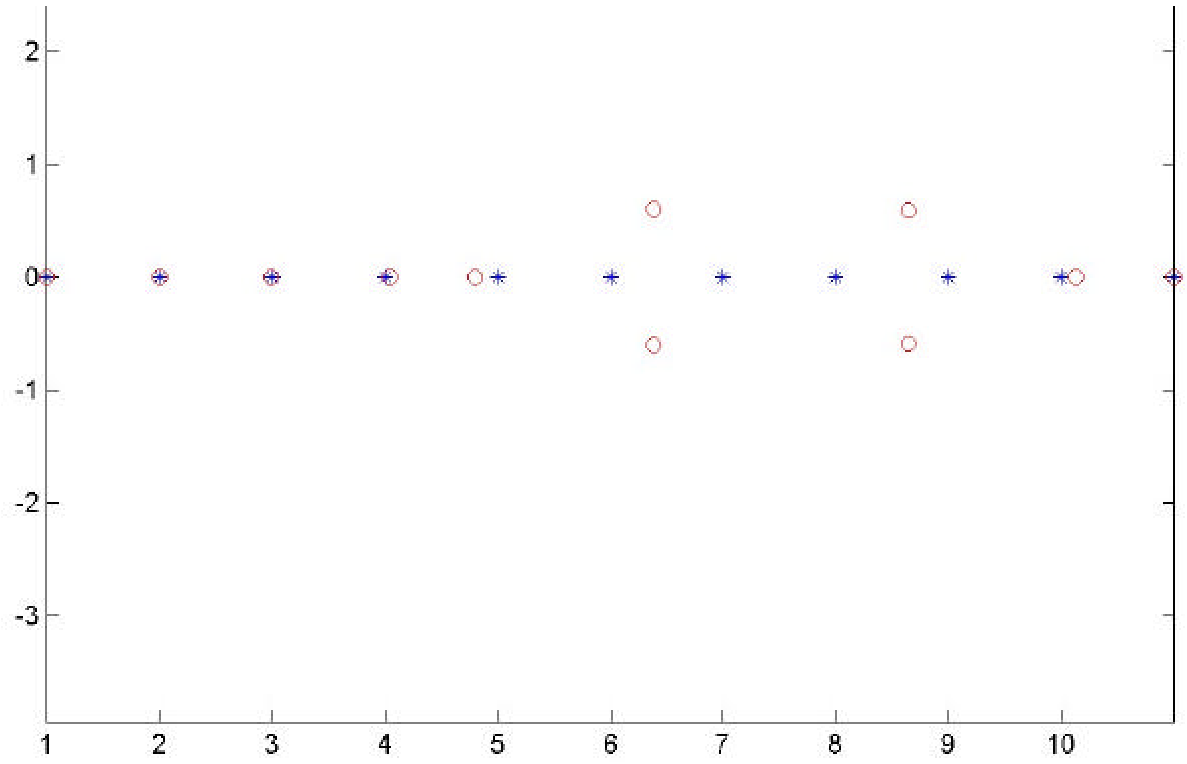
*=double

o=single

Accuracy:

single = 0.8

double = 0.8e-8



# Adaptive Precision

- Accuracy depends on precision
  - How much precision is needed for a specified accuracy?
  - Shouldn't the software figure this out for me?
- Adaptive multiple precision
  - Inputs:
    - Desired accuracy
    - Degrees of polynomials
    - Magnitude of the coefficients
  - Result:
    - Precision set to guarantee the desired accuracy

# Adaptive Formula

To get final accuracy of $e \leq 10^{-t}$, we need $P$ digits

$$P > t + \log_{10}\left(\left\|J^{-1}\right\|\Psi + \left\|z_*\right\|\right)$$

where

$$\Psi = d\sum_{i\in I}\left|c_i\right| \text{ bounds the function evaluation error}$$

$$d = \text{degree},\ \ c_i = \text{coefficients}$$

- There is a similar formula to guarantee path tracking and accuracy for multivariate systems
  - Bates, Sommese, Wampler, “Multiprecision path tracking”, preprint.

# Application to Wilkinson $W_{11}$

- Worst root is one near z=8
  - $||J^{-1}(8)|| = 1/30240$
  - $\Psi \sim 5 \times 10^9$
  - $||z||=8$
- Result: we need
  - $P > \tau + 5$
  - Single precision
    - P=8 → 3 good digits predicted, 1 obtained
  - Double precision
    - P=16 → 11 good digits predicted, 8 obtained

# Functionality: Top Level

Solve n polynomials f(z) in $\mathbf{C}[z_1,\ldots,z_N]$

- N and n not necessarily equal
- System f(z) might be a straight-line program
- We want all solutions
  - "solve" definition #2
- Higher-dimensional sets should be decomposed into irreducibles
- Solutions at infinity should cause no harm
- Should handle non-reduced components
- Should solve *ab initio* and *parameter* homotopies
- Should have a parallel version
- Should have the characteristics of
  - Reliable, Accurate, Fast → Adaptive Multiprecision
  - Modular, User Friendly
  - Let's discuss the modules…

# User Friendly & Flexibility of Usage

- What form should the software take to be most useful and user friendly?
- Possibilities
  - Black box with switches & configs (accuracy e, etc.)
    - File(s) in, file(s) out
  - Toolbox
    - Suite of C routines
    - Interface to X, where X = (Maple, Matlab, ?)
  - Scripting language
  - Engine for specialized wrappers
    - Example: kinematics with CAD/GUI interface
- Open discussion