Symbolic and Numerical Methods for Partial Differential Equations

Greg Reid, Jan Verschelde, Allan Wittkopf and Wenyuan Wu

email: reid@uwo.ca jan@math.uic.edu wittkopf@cecm.sfu.ca wwu25@uwo.ca

0. Background

- PDE occurs frequently in applications.
- We algorithmically study such systems, eg. to obtain global info about their solution spaces and construct formal power series solutions.
- Regard PDE theory from the point of view of geometry, so-called Jet Geometry of PDE, initiated by Sophus Lie and developed by Elie Cartan and others.

Apply a finite number of differentiations and eliminations (Diff. Elim. Methods)

Goal: to yield all missing constraints; the Zariski Closure of the union of all integral manifolds = algebraic varieties (local solvablility).

Advantages: eases the application symbolic and numeric solution techniques; enables the statement of EU Theorems for their solutions.

1. Motivation of Our Work

- Despite considerable progress in the exact case by Boulier, Hubert, Mansfield, Seiler etc. these methods are very expensive !
- This is a continuation of symbolic-numeric methods for differential systems begun in Wittkopf & Reid [1] , Reid, Smith & Verschelde [2].
- Numerical Algebraic Geometry[3] provides us a numerical way to study varieties. We replace these exact algebraic methods with a numerical method and use a combination of symbolic and numeric methods to make progress on this problem.

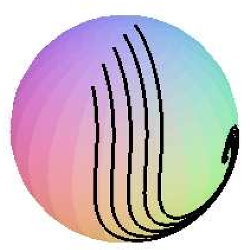
2. Formal Jet Theory

Consider q-th order PDE system $R = (R^1, ..., R^k) = 0$ with indep vars $x = (x_1, x_2, ..., x_n)$ and dep vars $u(x) = (u^1(x), u^2(x), ..., u^m(x))$. Denoting $\underset{r}{u}$ as the formal (jet) variables corresponding to r-th order partial derivatives of $u(x)$ the jet variety is:

$$V(R) := \{(x, u, \underset{1}{u}, ..., \underset{q}{u}) \in J^q : R(x, u, \underset{1}{u}, ..., \underset{q}{u}) = 0\},$$

$$\text{where } J^q = \mathbb{C}^{N_q}, N_q = n + m\binom{n+q}{q}.$$

Example (Tuoemela & Arponen): $V(R) = \{(x, u, u_x) : u_x^2 + u^2 + x^2 = 1\}$



All the extended solutions lying in $V(R)$ and for any regular point on $V(R)$ there exists an extended solution passing it. This property is called local solvablility.

References

[1] Wittkopf and Reid. **Fast differential elimination in C: The CDiffElim environment.** *Computer Physics Communications*, 139: 192–217, 2001.

[2] Reid, Smith, and Verschelde. **Geometric completion of differential systems using numeric-symbolic continuation.** *SIGSAM Bulletin* 36(2):1–17, 2002.

[3] Sommese and Wampler. **The Numerical solution of systems of polynomials arising in engineering and science.** World Scientific Press, Singapore, 2005.

3. A Symbolic Method: rifsimp

$[N, L, P]$ = ***rifsimp***$(S, \prec)$

Input: S, a system of polynomially nonlinear PDE and inequations over $\mathbb{Q}$, and a ranking $\prec$.

Output: a collection of cases each of the form [N,L,P] where L and N are a set of leading linear and leading nonlinear PDE resp.

- $\text{HD}(L)$ (leading derivatives) are all distinct, and no member of $\text{HD}(L)$ is a derivative of any other member
- N has no dependence on $\text{HD}(L)$
- P is a set of inequations with no dependence on $\text{HD}(L)$
- the integrability conditions of L reduced by L are in $\langle N \rangle$ (membership test !)
- $D(N)$ after reduction wrt L is in $\langle N \rangle$ (membership test !)
- An EU Theorem is provided for its output [4].

4. A Symbolic-numerical Method: HybridRif [5]

In rifsimp: membership tests are performed with rewriting techniques (Gröbner Bases, Triangular Sets). Instead we use Homotopy continuation [3][6], to detect new constraints by substituting suitable points into the presumed new constraints.

$[N, P, L, fail]$ = ***HybridRif***$(S, \text{ranking} = \prec, \epsilon)$

Input : S polynomially PDEs and a ranking $\prec$; a tolerance ϵ

Output: $[N, P, L, fail]$ where
N: leading nonlinear part; P: pivots (inequations)
L: leading linear part; `fail`: true if ill-conditioned

Step 1: differentiate the nonlinear part to generate some new equations

Step 2: use ***riflin*** to eliminate the high order vars and do the partition

Step 3: use ***MembershipTest*** to identify new eqns, if none then stop

Step 4: compute witness set of the intersection of new eqns and old nonlinear part by ***Intersect***. Return to **Step 1**.

riflin is a weakened rifsimp, only processing leading linear part.

5. Termination Conditions

HybridRif will fail when it encounters singularities (condition number of the Jacobian matrix at the witness points is very large) or ill-conditioned representations.
A standard Noetherian argument in [4], shows that the linear part must stabilize.
For the nonlinear part, in each iteration, if it generates some geometric new equations, the variety will become smaller. So, the nonlinear part will also stabilize.

6. Optimization

- Implementation of ***Intersect*** using Diagonal Homotopies
- As a system is prolonged (differentiated), # eqn. and #var. can grow dramatically. Removing such redundant polynomials by ***MembershipTest*** will not change the geometry of the solution set.
- Fix the value of the independent vars. to random numbers. Only focus on the meaningful solutions which are fibred over the space of independent variables X.

7. A Symbolic-Numeric Implementation

The method was tested and developed in Maple 10. It is implemented using PHCpack [6] and ***riflin***. The result is symbolic, in the form of $[N, L, P]$.

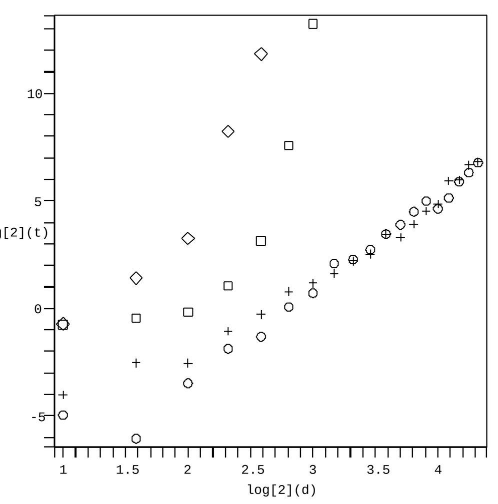
Example (Random first order ODE): $\mathbf{R(u_x, u) = 0}$

d	2	3	4	5	6	7	8	9
RAM: $\mathbb{Z}$	0.24	0.6	1.4	2.5	8.3	16.5	128.7	∞
RAM: $\mathbb{C}$	3.40	6.8	11.2	20.7	62.2	∞	∞	∞

rifsimp memory consumption (MB) applied to random polynomial ODE with integer(complex rational) coefficients. Here d = degree(R). RAM=∞ indicates that the memory was exhausted. total RAM : 512 MB.

d	2	5	8	11	14	17	20
RAM: $\mathbb{Z}$	1.5	1.8	2.0	2.1	2.2	2.3	2.5
RAM: $\mathbb{C}$	1.5	1.8	2.0	2.0	2.2	2.3	2.6

phc memory consumption (MB) applied to random ODE.



Time-Degree Statistics for a Random polynomial ODE.
Here d = degree(R) and t is the time to process: rifsimp(integ coeffs) □; rifsimp(comp coeffs) ◇; phc(integ coeffs) ○; phc(comp coeffs) +

8. Conclusion

- We give examples of better behavior than fully symbolic approach.
- Method has been applied to other systems with encouraging results.
- Future work includes treating systems with singular components, for example higher multiplicity (current version will fail).

For PDE systems with approx. input and a deeper understanding of approximate jet geometry please see Greg's talk on Friday (Oct. 27) !

More References

[4] Rust. **Rankings on derivatives for elimination algorithms and formal solvability of analytic partial differential equations.** *Ph.D. Thesis*, University of Chicago, 1998.

[5] Reid, Verschelde, Wittkopf and Wu. **Symbolic-Numeric Completion of Differential Systems by Homotopy Continuation.** *Proc. ISSAC 05*, 2005.

[6] Verschelde. **Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation.** *ACM Trans. Math. Softw.*, 25(2):251–276, 1999. Available at http://www.math.uic.edu/˜jan.

University of Western Ontario, Department of Applied Mathematics, London, ON. N6G 5B7, Canada