A Smooth and Algebraic Invariants of a Group Action: Local and Global Constructions.

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Abstract: We provide an algebraic formulation of the moving frame method for constructing local smooth invariants on a manifold under an action of a Lie group. This formulation gives rise to algorithms for constructing rational and replacement invariants. The latter are algebraic over the field of rational invariants and play a role analogous to Cartan's normalized invariants in the smooth theory. The algebraic algorithms can be used for computing fundamental sets of differential invariants.

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<tr><td colspan="2">Consider</td></tr>
<tr><td>Lie grp. $\mathcal{G} \curvearrowright M$ smooth manifold over $\mathbb{R}$
smooth, local, semi-regular action</td><td>Alg. grp. $\mathcal{G} \curvearrowright \mathcal{Z} \subset \mathbb{K}^n$ affine variety
rational action</td></tr>
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<tr><td colspan="2">Construct</td></tr>
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moving frame map $\rho : M \to \mathcal{G}$
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fundamental set for $\mathcal{F}(M)^{\mathcal{G}}$
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projection $\iota : \mathcal{F}(M) \to \mathcal{F}(M)^{\mathcal{G}}$</td><td>graph-section
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reduced Gröbner basis
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finite generating set for $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$
replacement inv. that are $\overline{\mathbb{K}(\mathcal{Z})^{\mathcal{G}}}$-tuples
$\Downarrow$
projection $\iota : \mathbb{K}(\mathcal{Z}) \to \overline{\mathbb{K}(\mathcal{Z})^{\mathcal{G}}}$</td></tr>
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Motivation

$$\boxed{\text{Smooth construction}} \quad \leftrightarrows \quad \boxed{\text{Algebraic construction}}$$

- $\longrightarrow$ new approaches for computation and study of algebraic invariants.

- $\longleftarrow$ computational methods for analyzing the structure of differential algebra of invariants.

Some Applications

- Symmetry reductions of differential and algebraic equations
- Equivalence problems (e.g. computer image recognition)

Smooth construction: $\mathcal{G}$ is a Lie group, M is a smooth manifold.

Local action $\mathcal{G} \curvearrowright M$ is a smooth map $g\colon \Omega \longrightarrow M$, where $\Omega \supset \{e\} \times M$ is open in $\mathcal{G} \times M$ s. t.

(i) $g(e, \bar{z}) = \bar{z}$, $\forall \bar{z} \in M$

(ii) $g(\bar{\mu}, g(\bar{\lambda}, \bar{z})) = g(\bar{\mu}\bar{\lambda}, \bar{z})$, for all $\bar{z} \in M$ and $\bar{\lambda}, \bar{\mu} \in \mathcal{G}$ s. t. $(\bar{\lambda}, \bar{z})$ and $(\bar{\mu} \cdot \bar{\lambda}, \bar{z})$ are in Ω.

(*Notation:* $g(\bar{\lambda}, \bar{z}) = \bar{\lambda} \cdot \bar{z}$)

Local smooth invariants: $f \in \mathcal{F}(\mathcal{U})$, where $\mathcal{U} \subset M$ is an open subset, s.t. $f(g(\bar{\lambda}, \bar{z})) = f(\bar{z})$, $\forall (\bar{\lambda}, \bar{z})$ in an open subset of Ω containing $e \times U$.

Infinitesimal criterion: $\mathbf{v}(f)(\bar{z}) = 0 \ \forall \bar{z} \in \mathcal{U}$, $\forall$ infinitesimal generators $\mathbf{v}$ of $\mathcal{G} \curvearrowright M$.

($\mathcal{G}$ *connected,* $\mathbf{v}(f)(\bar{z}) = 0$, $\forall \bar{z} \in M \Rightarrow f$ *is global invariant*)

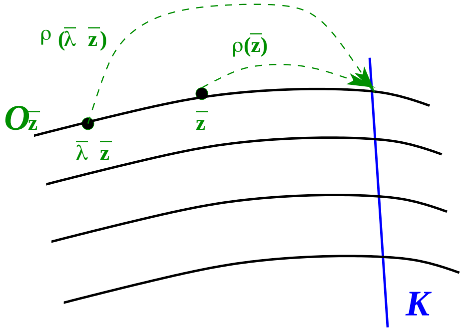
Smooth construction:

Local cross-section $\mathcal{K}$ on $\mathcal{U} \subset M$ is a submanifold, s. t.

- $T_{\bar{z}}\mathcal{K}| \bigoplus T|_{\bar{z}}\mathcal{O}_{\bar{z}} = T|_{\bar{z}}M, \quad \forall \bar{z} \in \mathcal{K}$ (transversality condition)
- $\mathcal{K}$ intersects each connected component of $\mathcal{O}_{\bar{z}} \cap \mathcal{U}$ at the unique point.

$\mathcal{G} \curvearrowright M$ semi. reg. (i.e. $\exists$s, $\forall \bar{z} \in M \dim \mathcal{O}_{\bar{z}} = s$) $\xRightarrow{\text{Frobenius Thm.}}$ $\forall \bar{z} \in M \; \exists \mathcal{K} \ni \bar{z}$.

Moving frame map (Fels, Olver (1999)) defined by $\rho(\bar{z}) \cdot \bar{z} \in \mathcal{K}$



$\mathcal{G} \curvearrowright M$ free $\Rightarrow \rho$ is smooth, $\mathcal{G}$-equivariant:

$$\rho(\bar{\lambda} \cdot \bar{z}) \cdot (\bar{\lambda} \cdot \bar{z}) = \rho(\bar{z}) \cdot \bar{z} \xRightarrow{\text{freeness}} \rho(\bar{\lambda} \cdot \bar{z}) = \rho(\bar{z}) \cdot \bar{\lambda}^{-1}$$

Smooth construction: Invariantization: $\iota: \mathcal{F}(\mathcal{U}) \to \mathcal{F}^{\mathcal{G}}(\mathcal{U})$

- Defined via the moving frame map ρ (Fels, Olver (1999))

$$\iota f(\bar{z}) = f(\rho(\bar{z}) \cdot \bar{z})$$

ρ is non-constructive – existence relies on the implicit function theorem

- Defined as restriction to a cross-section $\mathcal{K}$

$$\forall \bar{z} \in \mathcal{U} \quad \iota f(\bar{z}) = f(\bar{z}_0), \text{where } \bar{z}_0 = \mathcal{O}^0_{\bar{z}} \cap \mathcal{K}.$$

  i. $\mathcal{K}$ is constructive – $\forall \bar{z} \in \mathcal{Z}$ one can construct c.s. defined as a level set of $n-s$ coord. functions.

  ii. freeness is not required.

THEOREM 1.

- If the action is free both definitions are equivalent
- ιf is the unique smooth local invariant s.t. $\iota f|_{\mathcal{K}} = f|_{\mathcal{K}}$

Smooth construction: Normalized and fundamental invariants

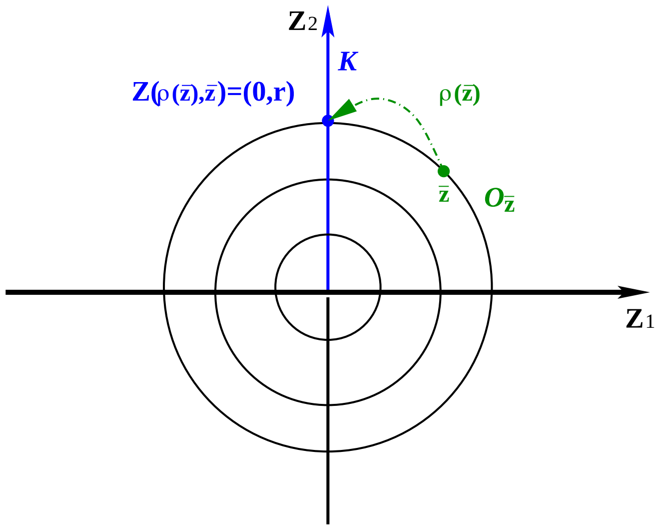
DEFINITION.
If $z = (z_1, \dots, z_n)$ are coordinate functions on $\mathcal{U}$ then $\iota z_1, \dots, \iota z_n$ are called normalized invariants

THEOREM 2.

- Invariantization: If $f \in \mathcal{F}(\mathcal{U})$ then $\iota f(z_1, \dots, z_n) = f(\iota z_1, \dots, \iota z_n) \in \mathcal{F}(\mathcal{U})^{\mathcal{G}}$
- Replacement: If $f \in \mathcal{F}(\mathcal{U})^{\mathcal{G}}$ then $f(z_1, \dots, z_n) = f(\iota z_1, \dots, \iota z_n) = \iota f(z)$.
- Fundamental set: Let $\dim \mathcal{O}_{\bar{z}} = s, \forall \bar{z} \in \mathcal{U}$, then $\{\iota z_1, \dots, \iota z_n\}$ contains $n - s$ functionally independent local inv., s.t. any $f \in \mathcal{F}(\mathcal{U})^{\mathcal{G}}$ can be locally expressed as function of these invariants in a unique way.

Smooth Example 1: $SO(2) = \left\{ \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \right\} \curvearrowright M = \mathbb{R}^2/\{(0,0)\}$

Action: $Z_1 = \cos(\phi)\, z_1 - \sin(\phi)\, z_2, \quad Z_2 = \sin(\phi)\, z_1 + \cos(\phi)\, z_2$



Cross-section:

$\mathcal{K} = \{(z_1, z_2) | z_1 = 0,\ z_2 > 0\} \Rightarrow$

Moving frame map is defined by $Z_1 = 0$

i.e. $\rho(z) = \begin{pmatrix} \frac{z_2}{\sqrt{z_1^2+z_2^2}} & -\frac{z_1}{\sqrt{z_1^2+z_2^2}} \\ \frac{z_1}{\sqrt{z_1^2+z_2^2}} & \frac{z_2}{\sqrt{z_1^2+z_2^2}} \end{pmatrix}$

Invariantization: $f(z_1, z_2) \in \mathcal{F}(M) \rightarrow$
$\iota f(z) = f(\rho(z) \cdot z) = f(0, \sqrt{z_1^2 + z_2^2}) \in \mathcal{F}^{\mathcal{G}}$

Normalized invariants: $\iota z_1 = 0,$

$\iota z_2 = \sqrt{z_1^2 + z_2^2} = r$ (fundamental invariant)

Replacement property: $f \in \mathcal{F}(M)^{\mathcal{G}} \Rightarrow f(z) = f(\iota z_1, \iota z_2) = f(0, r)$ i.e. $z_1^2 + z_2^2 = (\iota z_1)^2 + (\iota z_2)^2 = 0^2 + r^2$

Restrictions to the cross-section: $\iota f|_{\mathcal{K}} = f|_{\mathcal{K}} = f(0, r)$

Smooth Example 2: $SE(2) = SO(2) \ltimes \mathbb{R}^2 \curvearrowright$ on plane curves $y = y(x)$:

Action: $X = \cos(\phi)x - \sin(\phi)y + a, \quad Y = \sin(\phi)x + \cos(\phi)y + b$

Prolongation to derivatives up to second order:

$$Y_1 = \frac{\sin(\phi) + \cos(\phi)y_1}{\cos(\phi) - \sin(\phi)y_1}, \quad Y_2 = \frac{y_2}{(\cos(\phi) - \sin(\phi)y_1)^3}$$

Cross-section: $\mathcal{K} = \{(x, y, y_1, y_2) | x = 0,\ y = 0,\ y_1 = 0, y_2 > 0\} \subset \mathbb{R}^4$

Moving frame map is defined by: $X = 0,\ Y = 0,\ Y_1 = 0 \Rightarrow$

$$\cos\phi = \frac{1}{\sqrt{y_1^2+1}}, \ \sin\phi = -\frac{y_1}{\sqrt{1+y_1^2}}, \quad a = -\frac{x+y_1y}{\sqrt{1+y_1^2}}, \quad b = \frac{y_1x-y}{\sqrt{1+y_1^2}}.$$

Normalized invariants: $\iota x = 0,\ \iota y = 0, \iota y_1 = 0,$

$\iota y_2 = \frac{y_2}{(1+y_1^2)^{3/2}}$ is curvature –differential invariant.

Algebraic construction: $\mathcal{G}$ is an alg. group, $\mathcal{Z}$ is an affine variety.

Rational action $\mathcal{G} \curvearrowright \mathcal{Z}$ is a rational map g: $\mathcal{G} \times \mathcal{Z} \to \mathcal{Z}$ s. t.

(i) $g(e, \bar{z}) = \bar{z}$, $\forall \bar{z} \in \mathcal{Z}$

(ii) $g(\bar{\mu}, g(\bar{\lambda}, \bar{z})) = g(\bar{\mu}\,\bar{\lambda}, \bar{z})$, $\forall \bar{\lambda}, \bar{\mu} \in \mathcal{G}$ and $\forall \bar{z} \in \mathcal{Z}$ s.t. $g(\bar{\lambda}, \bar{z})$ and $g(\bar{\mu}\,\bar{\lambda}, \bar{z})$ are defined.

(*Notation:* $g(\bar{\lambda}, \bar{z}) = \bar{\lambda} \cdot \bar{z}$)

Rational invariants: $f \in \mathbb{K}(\mathcal{Z})$ s.t. $f(g(\bar{\lambda}, \bar{z})) = f(\bar{z})$, $\forall \bar{z} \in \mathcal{Z}$ and $\forall \bar{\lambda} \in \mathcal{G}$ s.t. $g(\bar{\lambda}, \bar{z})$ is defined.

Algebraic construction: Notation: $\overline{S}$ is Zariski closure of set S

Graph of the action $\mathcal{O} = \overline{\{(\bar{z}, \bar{z}') \subset \mathcal{Z} \times \mathcal{Z} | \exists \bar{\lambda} \in \mathcal{G} : \ \bar{z}' = \bar{\lambda} \cdot \bar{z}\}}$ ↔ ideal: $O \subset \mathbb{K}[\mathcal{Z} \times \mathcal{Z}]$
extension: $\boxed{O^e \subset \mathbb{K}(\mathcal{Z})[\mathcal{Z}]}$

Orbit: $\mathcal{O}_{\bar{z}} = \overline{\{\bar{z}' \in \mathcal{Z} | \exists \bar{\lambda} \in \mathcal{G} : \ \bar{z}' = \bar{\lambda} \cdot \bar{z}\}}$↔ ideal: $O_{\bar{z}} \subset \mathbb{K}[\mathcal{Z}]$

Cross-section of degree d: an irreducible subvariety $\mathcal{K} \subset \mathcal{Z}$ s.t. $\mathcal{O}_{\bar{z}} \cap \mathcal{K}$ consists of d simple points $\forall \bar{z}$ in a dense subset of $\mathcal{Z}$ (transversality condition)

$$\uparrow$$

ideal: K is prime, s.t. $codim K = \max_{\bar{z}} \dim \mathcal{O}_{\bar{z}} = s$ and $\boxed{I^e = O^e + K \subset \mathbb{K}(\mathcal{Z})[\mathcal{Z}]}$ is radical zero-dimensional (transversality cond.)

Graph-section: $\mathcal{I} = \{(\bar{z}, \bar{z}') \subset \mathcal{Z} \times \mathcal{K} | \exists \bar{\lambda} \in \mathcal{G} : \ \bar{z}' = \bar{\lambda} \cdot \bar{z}\}$↔ ideal: $I = O + K \subset \mathbb{K}[\mathcal{Z} \times \mathcal{Z}]$

Algebraic construction: Generating set of $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$.

THEOREM 3:
Coeff. of a reduced Gröbner basis Q of either O^e or I^e generate $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$.

Ideas of the proof:

- If $(\bar{z}, \bar{z}') \in \mathcal{O} \Rightarrow (\bar{\lambda} \cdot \bar{z}, \bar{z}') \in \mathcal{O}, \forall \bar{\lambda} \in \mathcal{G}$. Hilbert's Nullstellensatz & uniqueness of the reduced $Q \longrightarrow$ coeff. of Q are in $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$
- Rewriting algorithm to express $f(z) = \frac{p(z)}{q(z)} \in \mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ in terms of generators by computing normal forms of p and q w.r.t. Q.

Previous work. Rosenlicht (1956): $\forall$ subset set of $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ that separates orbits generates $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$; coeffs. of Chow form of O^e have this property.

Vinberg, Popov (1989): if coeff. of a generating set of O^e are in $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$, then they generate $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$; $\exists$ such generating set.

Beth, Müller-Quade (1999): rewriting algorithm for linear actions.

Contribution: Simple algorithm applicable to both O^e and I^e. $\dim I^e = 0 \Rightarrow$ computational advantage.

Algebraic construction: Replacement invariants

LEMMA. Let Q be a reduced Gröbner basis of the graph-section ideal $I^e = (O^e + K) \subset \mathbb{K}(\mathcal{Z})[\mathcal{Z}]$. Then

- Q is a reduced Gröbner basis of $I^{\mathcal{G}} = I^e \cap \mathbb{K}(\mathcal{Z})^{\mathcal{G}}[\mathcal{Z}]$.
- $I^{\mathcal{G}} \subset \mathbb{K}(\mathcal{Z})^{\mathcal{G}}[\mathcal{Z}]$ is zero-dimensional and prime.
- If c.-s. $\mathcal{K}$ has degree d then $I^{\mathcal{G}}$ has d simple $\overline{\mathbb{K}(\mathcal{Z})^{\mathcal{G}}}$-zeros $\xi^{(i)} \colon [\overline{\mathbb{K}(\mathcal{Z})^{\mathcal{G}}}]^n \to \mathcal{Z}$, $i = 1..d$, (called replacement invariants associated with $\mathcal{K}$.)

THEOREM 4. Let $\xi^{(i)}$, $i = 1..d$, be replacement invariants associated with $\mathcal{K}$ of degree d. Then

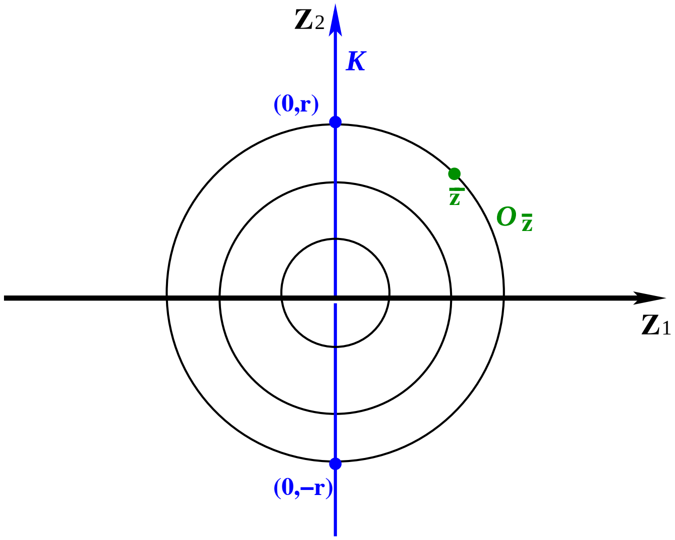
- $f(z) \in \mathbb{K}(\mathcal{Z})^{\mathcal{G}} \Rightarrow f(z) = f\left(\xi^{(i)}\right)$ for $i = 1..d$. (replacement)
- $\mathbb{K}(\mathcal{K}) \cong \mathbb{K}(\xi^{(i)})$, $i = 1..d$ is the extension of degree d of $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$.
- $\iota_i \colon \mathbb{K}[\mathcal{Z}]_{\mathcal{K}} \to \overline{\mathbb{K}(\mathcal{Z})^{\mathcal{G}}} : \iota_i f = f\left(\xi^{(j)}\right)$ for $j = 1..d$ is computable projection (invariantization)

COROLLARY. If $d = 1$ there is a unique rational replacement invariant $\xi \colon [\mathbb{K}(\mathcal{Z})^{\mathcal{G}}]^n \to \mathcal{Z}$, that provides a generating set of $\mathbb{K}(\mathcal{Z})^{\mathcal{G}}$ (cf. Vinberg, Popov (1989)):

$$\mathbb{K}(\mathcal{Z})^{\mathcal{G}} \cong \mathbb{K}(\xi) \cong \mathbb{K}(\mathcal{K})$$

Algebraic Example 1: (rotation) $SO(2) \curvearrowright \mathbb{K}^2$

Group ideal: $G = (\lambda_1^2 + \lambda_2^2 - 1) \subset \mathbb{K}[\lambda_1, \lambda_2]$
Action ideal: $J = (Z_1 - (\lambda_1 z_1 - \lambda_2 z_2),\, Z_2 - (\lambda_2 z_1 + \lambda_1 z_2)) + G$



Graph ideal: $O = J \cap K[\mathcal{Z} \times \mathcal{Z}]$
$Q = \{Z_1^2 + Z_2^2 - (z_1^2 + z_2^2)\}$ is rGB for O and O^e.
Cross-section ideal (of degree 2):
$K = (Z_1)$
Graph-section ideal: $I = O + K$
$Q = \{Z_1, Z_2^2 - (z_1^2 + z_2^2)\}$ is rGB for I and I^e

Replacement invariants: are $\overline{\mathbb{K}(\mathcal{Z})^{\mathcal{G}}}$ roots of I^e: $(\xi_1^{\pm}, \xi_2^{\pm}) = (0, \pm r)$
Replacement property:
$f \in \mathbb{K}(\mathcal{Z})^{\mathcal{G}} \Rightarrow f(z) = f(\xi^{\pm})$ i.e. $z_1^2 + z_2^2 = (\xi_1^{\pm})^2 + (\xi_2^{\pm})^2$

Algebraic Example 2: $SE(2) \curvearrowright$ plane curves.

Notation: $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $z = (x, y, y_1, y_2)$, $Z = (X, Y, Y_1, Y_2)$

Group ideal: $G = (\lambda_1^2 + \lambda_2^2 - 1) \subset \mathbb{K}[\lambda]$

Action ideal:

$$\begin{aligned} J \;=\; & \Big(X - (\lambda_1 x - \lambda_2 y + \lambda_3), Y - (\lambda_2 x + \lambda_1 y + \lambda_4), \\ & Y_1 - \frac{\lambda_2 + \lambda_1 y_1}{\lambda_1 - \lambda_2 y_1}, Y_2 - \frac{y_2}{(\lambda_1 - \lambda_2 y_1)^3}\Big) + G \subset \mathbb{K}[\lambda, z, Z] \end{aligned}$$

Cross-section ideal: $K = (X, Y, Y_1)$

Graph-section ideal: $\left\{X, Y, Y_1, Y_2^2 - \frac{y_2^2}{(1+y_1^2)^3}\right\}$ is rGB for $I = (J + K) \cap \mathbb{K}[z, Z]$, $I^e \in \mathbb{K}(z)[Z]$ and $I^G \in \mathbb{K}(\mathcal{Z})^{\mathcal{G}}[Z]$

Replacement Invariants: $\overline{\mathbb{K}(\mathcal{Z})^{\mathcal{G}}}$-roots if I^e: $\xi_1^{\pm} = (0, 0, 0, \pm\kappa)$, where $\kappa = \frac{y_2}{(1+y_1^2)^{3/2}}$ is curvature.