Representing NP-Complete problems as Polynomial Ideals: Degree Growth for the Hilbert Nullstellensatz and Properties of their Gröbner bases

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1 Key Points

Systems of polynomial equations over the complex numbers can be used to characterize combinatorial decision problems. From the point of view of computer algebra and symbolic computation these are interesting polynomial systems because:

1) They are provably-hard polynomial systems, because solving them is as hard as solving NP-complete problems. Thus the structure of their Gröbner bases is expected to show extreme behavior (e.g., degree growth, bad running times).

2) Unless NP = coNP, there must exist infinite instances of these infeasible systems whose Hilbert Nullstellensatz certificate grows exponentially.

In this preliminary report we look at various polynomial ideal encodings, their Gröbner bases, and their Hilbert Nullstellensatz certificates for graph $k$-colorability.

2 Representation 1 of Graph Coloring: Degree $k$ Polynomial Ideal (D. Bayer)

Given a graph $G$ and an integer $k$, we construct the following ideal:

- vertex polynomials: For every vertex $i = 1, \ldots, n,$
  \[ x_i^k - 1 = 0 \]

- edge polynomials: For every edge $(i, j) \in E(G),$
  \[ \frac{x_i - x_j}{x_i - x_j} = x_i^k - x_j^k \]
  \[ \ldots + x_{i+p}^k - x_{j+p}^k = 0 \]

- Theorem: Let $G$ be a graph, $k$ an integer, encoded as vertex and edge polynomials. Then this system of equations has a solution if and only if $G$ is $k$-colorable.

3 Representation 2 of Graph Coloring: Degree 2 Polynomial Ideal

- vertex partition polynomials: For every vertex $i = 1, \ldots, n,$
  \[ x_{i,p} + x_{i,p+1} + \cdots + x_{i,k} = 1 \]

- edge partition polynomials: For every edge $(i, j) \in E(G)$ and every partition $p = 1, \ldots, k$
  \[ x_{i,p}x_{j,p} = 0 \]

- variable constraint polynomials: For every variable $x_{i,p}$
  \[ x_{i,p}(x_{i,p} - 1) = 0 \]

- Theorem: Let $G$ be a graph, $k$ an integer, encoded as vertex partition, edge partition and variable constraint polynomials. Then this system of equations has a solution if and only if $G$ is $k$-colorable.

4 Gröbner Bases: Representation 1 vs. Representation 2

Comparing the running times for finding Gröbner bases of the degree 2 and 3 ideals for the not 3-colorable odd wheels.

5 Testing Graphs for 3-colorability Using CoCoA

| Graph | $|V|$ | $|E|$ | time |
|-------|------|------|------|
| 363-odd wheel | 3 | 14 | 124 sec |
| 364-odd wheel | 3 | 14 | 124 sec |
| (6,2)-Kneser | 3 | 14 | 0.04 sec |
| (7,2)-Kneser | 3 | 21 | 0.15 sec |
| (7,3)-Kneser | 3 | 35 | 0.99 sec |
| (8,2)-Kneser | 3 | 28 | 0.33 sec |
| (8,3)-Kneser | 3 | 56 | 1.09 sec |
| (8,5)-Kneser | 3 | 94 | 1.09 sec |
| (9,4)-Kneser | 3 | 126 | 5.71 sec |
| (10,5)-Kneser | 3 | 60 | 3.0 sec |
| (11,2)-Kneser | 3 | 55 | 16 sec |

Note that deciding a not-$k$-colorable instance runs very quickly, but deciding a $k$-colorable instance runs slowly, because computing the Gröbner basis is equivalent to counting the number of $k$-colorings.

6 NP, coNP and the Nullstellensatz

- Lemma: If NP $\neq$ coNP, then there must exist an infinite family of graphs such that the minimum-degree Nullstellensatz for not $k$-colorability grows with respect to the input size of the underlying graph.

- Challenge: Construct the infinite family of graphs where the minimum Nullstellensatz degree grows exponentially.

How can we find such Hilbert Nullstellensätze explicitly?

7 Odd Wheels and the Nullstellensatz

- Theorem: The minimum degree Nullstellensatz for odd-wheels is four.

- Proof Sketch:

We derive a certificate for the $(n+1)$-odd wheel from the $n$-th odd wheel, by taking a very particular syzygy on some of the terms from the $n$-th odd wheel not 3-colorable certificate.

8 Some Experimental Results

- Lemma: The Jin graph has a Nullstellensatz certificate of degree $\geq 7$.

Note: All other graphs tried have degree 4 (e.g., $K_n$ with $n \geq 1$ has degree 4).

9 Searching for Hilbert Nullstellensatz degrees...

Nullstellensatz: if $V = \{f_0, \ldots, f_n\}$, then $1 = \sum_{i=0}^n a_i f_i$.

\[ 1 = (c_1 + c_2 + c_3 + c_4)(y_1^2 - 1) + (c_1 + c_2 + c_3 + c_4)(y_1^2 - 1) + (c_1 + c_2 + c_3 + c_4)(y_1^2 - 1) + (c_1 + c_2 + c_3 + c_4)(y_1^2 - 1) + (c_1 + c_2 + c_3 + c_4)(y_1^2 - 1) + (c_1 + c_2 + c_3 + c_4)(y_1^2 - 1) + (c_1 + c_2 + c_3 + c_4)(y_1^2 - 1) + (c_1 + c_2 + c_3 + c_4)(y_1^2 - 1) + (c_1 + c_2 + c_3 + c_4)(y_1^2 - 1) \]

Try a degree for the $c_i$ polynomials, and construct a large-scale sparse linear system of equations. If infeasible, try a larger degree for $c_i$. Note: deg$c_i$ cannot exceed known upper bounds for Hilbert Nullstellensatz.

10 Open Questions/Work in Progress

- C++ implementation of the Nullstellensatz degree search algorithm, with special improvements for reducing number of unknowns.

- Conduct identical analysis and experiments for other NP-Complete problems, such as Hamiltonian cycle, Stable Set and $3$-SAT.