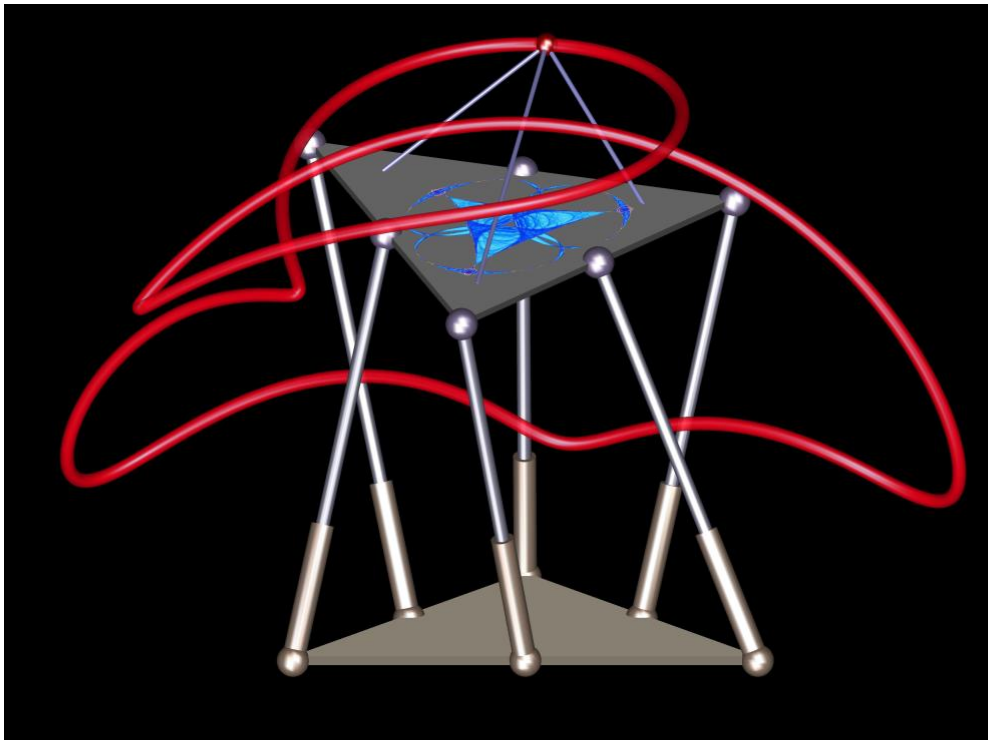
IMA program on Applications of Algebraic Geometry

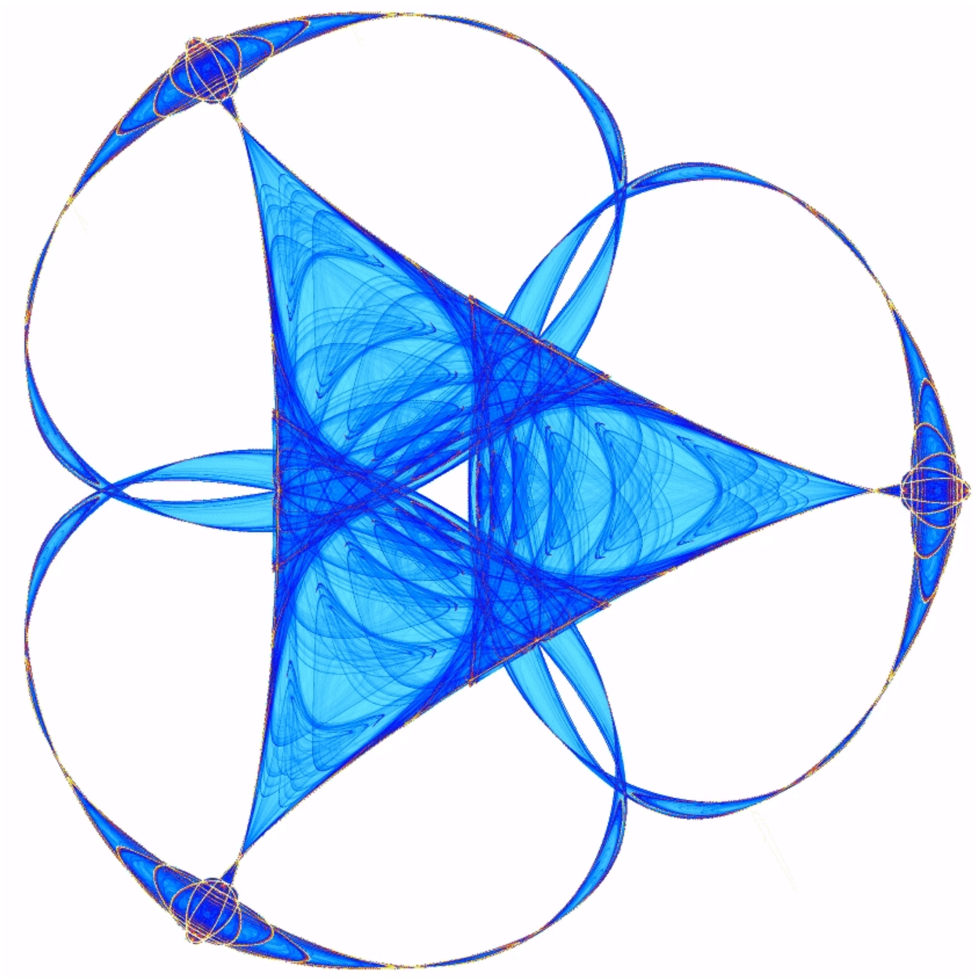
Frank Sottile

sottile@math.tamu.edu

http://www.math.tamu.edu/~sottile



Why algebraic geometry at the IMA ?



Why algebraic geometry at the IMA ?

- We solve equations.

- The objects in our subject appear in many different applications.

  The positive part of a toric variety, (my subject) occurred in all four workshops I attended. (once my fault)

  The applications included: Bounds on real solutions to polynomial equations, algebraic statistics, phylogenetic analysis, dynamics of chemical reaction networks, computational complexity, and geometric modeling.

People

The IMA brings people together for sustained, high-level interaction.

For example, Charles Wampler, Technical Fellow at GM Research and Development Center, took a sabbatical while his collaborator, Andrew Sommese of Notre Dame, had a concurrent sabbatical.

Many, many others enjoyed productive, medium-to-long-term visits.

Here is what Charles had to say about his visit:

"My research visit to the IMA last fall (mid-September through October 2006) was very productive. The workshops on algebraic geometry drew an extremely strong contingent of researchers in the field, opening up many possible avenues of collaboration. In between the workshops, my office was constantly busy with visitors discussing new work and possible extensions. I only wish my stay had been longer to allow more time to contemplate all of the stimulating input. It will take years for me and my collaborators to follow up on all the leads established in that short time."

— Charles Wampler

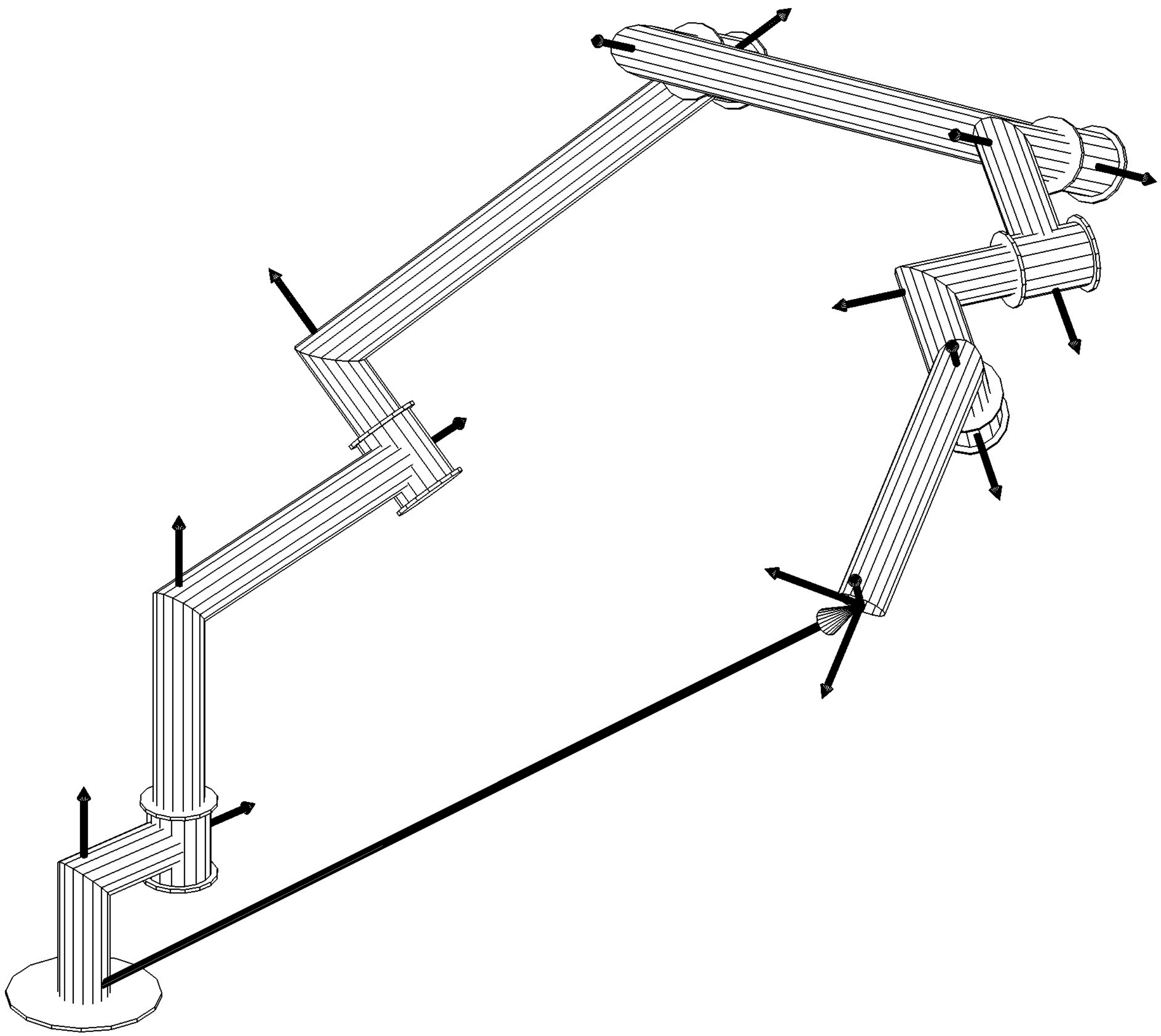
Bertini

IMA postdoc Daniel J. Bates, Notre Dame graduate student Jon Hauenstein, Sommese, and Wampler took advantage of their time at the IMA to launch Bertini, a software package for numerical algebraic geometry.

This is an emerging field that uses numerical methods to study algebraic varieties. Wampler's field of kinematics has driven much of the development in this field.

Among other features, Bertini can solve equations, big equations.

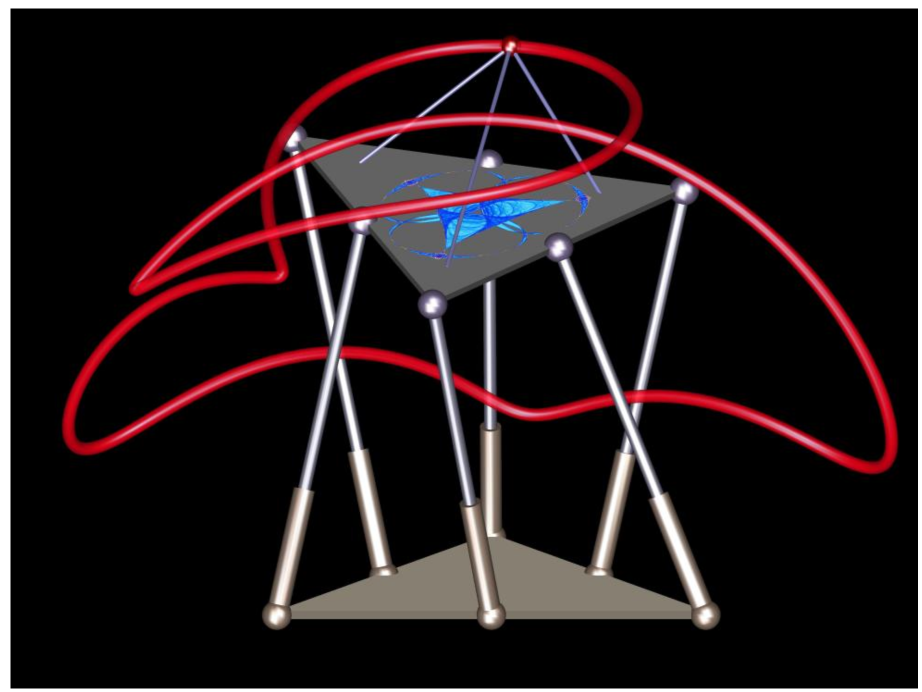
Mt. Everest of Kinematics: Six-revolute, serial link robot



During the talk, I ran Bertini on this famous continuation problem — this is an example from the Bertini web page. It took 2 minutes to solve this problem in inverse kinematics.

(CAE AW139 helicopter simulator)

Degenerate Stewart Platforms



Optimization

One important class of problems in optimization are semi-definite programs. A typical program seeks to minimize $C \bullet X$ subject to $A_k \bullet X = b_k$, $k = 1, \ldots, m$ and $X \succeq 0$.

Here, C, A_k, X are symmetric matrices, b_k are scalars, $A \bullet B :=$ trace(AB), the standard inner product on symmetric matrices, and $X \succeq 0$ means that X must lie in the cone of positive semidefinite (psd) matrices.

"Optimization over a linear section of the psd cone."

Semi-definite programs (SDP) are efficient to solve. This subject has found many recent applications through its link to real algebraic geometry via sums of squares.

Many hard optimization problems enjoy well-behaved relaxations where positivity of a polynomial (which is hard) is replaced by the condition that the polynomial is a sum of squares (an easy SDP).

LMI representation of convex sets

Over which convex sets can we optimize?

SDP can optimize over those S which have a Linear Matrix Inequality (LMI) representation

$$S = \{x \in R^n \mid A_0 + x_1 A_1 + \cdots + x_m A_m \succeq 0\},$$

where A_i are symmetric matrices.

S must be convex and defined by polynomial inequalities, but this is not sufficient. The following set does not have such a representation.

$$\{(x, y) \in R^2 \mid x^4 + y^4 \leq 1\}$$

LMI representation of convex sets (continued)

The set $\{(x, y) \in R^2 \mid x^4 + y^4 \leq 1\}$ is however, the projection of an LMI set, which is good enough for optimization.

Recently, IMA postdoc Jiawang Nie and Bill Helton of UCSD completed a preprint "SDP representations of convex sets" in which they give very general sufficient conditions on a convex set to guarantee such a semidefinite representation.

While the problem of characterizing such sets remains open, this represents a major advance on this important structural question.

Biology and Statistics

Sturmfels (IMA Board member and a lead organizer of the year) and Pachter (organizer of March workshop) have championed an approach to computational biology and algebraic statistics using tools from tropical geometry, polytope theory, toric varieties, and commutative algebra.

These new tools are beginning to make an impact in the biology community.

With Beerenwinkel, they introduced the genotope, a polytope encoding the genotypes in a population, as a tool to find structural information about genetic variation within populations.

Fitness landscape

The fitness landscape encodes the fitness of different genotypes. In their formulation, it is a graph, or lifting, of the genotope whose convexity reveals comparisons between the relative fitness of different combinations of genotypes.

With coauthors Elena and Lenski they show this additional information is useful to understand the fitness of different genotypes of $E.\ coli$ cultured by Lenski.

In May 2007, this was the most downloaded paper from the journal BMC Evolutionary Biology with over 2,000 hits in its first three weeks on line. See: `www.ima.umn.edu/nuggets/evogeo.html`.

A feeling for the unity of this subject can be inferred from the following.

The fitness landscape is a lifting of the genotope, and finding such liftings with good convexity properties is crucial to the efficient implementation of numerical continuation methods that were mentioned in the beginning of my discussion.