1. Consider an semi-infinite array of compartments with only the first one receiving injected current. Can you prove that

$$V_1/I_1 = \frac{R}{1 + \frac{R}{R_{couple}}(1-\mu)}$$

where

$$\mu = 1 + z - \sqrt{z^2 + 2z}, \quad z = \frac{R}{2R_{couple}}.$$

(Here is a hint. Show that the steady state voltages satisfy:

$$V_{j+1} - 2(1+z)V_j + V_{j-1} = 0$$

except for $j = 1$. The general solution to this difference equation is just $V_j = A\mu_1^j + B\mu_2^j$ where $\mu_{1,2}$ are roots to $\mu^2 - 2(1+z)\mu + 1 = 0$. One of these roots, say, μ_2, is greater than 1 so that as $j \to \infty$ you better choose $B = 0$. Choose A so that the correct equation for V_1 holds:

$$0 = -\frac{V_1}{R} + \frac{V_2 - V_1}{R_{couple}} + I.$$

)

2. Consider a general N-compartment model for a passive neuron with current injected into some or all of the compartments. This will obey the following differential equations:

$$C_j \frac{dV_j}{dt} = I_j + \sum_k g_{jk}(V_k - V_j) - g_{L,j}(V_j - V_{leak})$$

Suppose that $g_{jk} \geq 0$, $C_j > 0$, $g_{L,j} > 0$. Prove that there is a unique equilibrium point to this and that it is asymptotically stable. (Hint: This is a diagonally dominant system.)

3. From the Nernst-Planck equation:

$$I = -\left(uzRT\frac{\partial[C]}{\partial x} + uz^2F[C]\frac{\partial V}{\partial x}\right) = 0.$$

deduce the Nernst equation by setting the current to zero. Then assume the potential gradient is constant, $\partial V/\partial x = V_m/l$. Let

$$[C](0) = [C]_{in}, \qquad [C](l) = [C]_{out}.$$

Find the current such that the NPE is satisfied with the constant field assumption and these two boundary conditions.

4. Let

$$u' = f(u,v), \qquad v' = g(u,v)$$

where f, g are continuously differentiable. Suppose $f_v g_u > 0$ in the plane. Show there are no periodic solutions. (Hint: In fact prove that if $u(t)$ is a solution, then $u(t)$ can never have a local minumum follwing a local maximum.

5. Consider the Izhikevich model

$$V' = I + V^2 - u, \qquad u' = a(bV - u)$$

along with the reset conditions. Analyze the equilibria and find (i) the curve of saddle-nodes; (ii) the curve of Hopf bifurcations; (iii) and the Takens-Bogdanov point (double zero eigenvalue) treating I, a as parameters. As a very difficult bonus, prove the Hopf bifurcation is sub-critical.

6. How many limit cycles can you find in the nice symmetric model:

$$u' = -u + \tanh(a[u - bv]), \qquad \tau v' = -v + \tanh(c[u - dv])$$