Turbulent Solutions of the Stochastic Navier-Stokes Equation

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Outline

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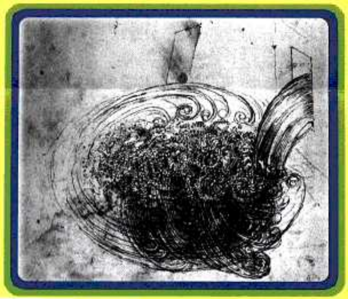
Leonardo's Observations



"Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the *principal current*, the other to the *random and reverse motion*."

Outline

How Does the Navier-Stokes Equation Become a Stochastic PDE?

- The flow satisfies the Navier-Stokes Equation

$$u_t + u \cdot \nabla u = \nu \Delta u + \nabla \{\Delta^{-1}[trace(\nabla u)^2]\}$$

- The initial data $u(x,0) = u_0(x)$ is large,
- This makes the initial value problem ill-posed
- The flow will become turbulent and the small noise ubiquitous in all flow will grow

The Appropriate Theory for Turbulence is a Statistical Theory

- In turbulent flow deterministic theory does not make any sense
- Instead we want a statistical theory
- This means that we want to be able to predict the statistical quantities associated with the flow
- One such statistical quantity is the mean $\overline{u}$ of the flow velocity

Why an Invariant Measure?

- In experiments and simulations it is an ensamble average

$$\overline{u} = \langle u \rangle$$

- Mathematically speaking the mean is an expectation, if ϕ is any bounded function on H

$$E(\phi(u)) = \int_H \phi(u) d\mu(u) \tag{1}$$

- We must prove the existence of a unique invariant measure to make mathematical sense of the mean $\overline{u}$ and other statistical quantities

Outline

Noise-driven Instabilities

- Let $U(x_1)j_1$ denote the mean flow (Leonardo's principal current), taken to be in the x_1 direction
- If $U' < 0$ the flow is unstable
- The largest wavenumbers (k) can grow the fastest
- Linearize the equation about the initial flow $U = U_0 j_1 + U'(x_1, -\frac{x_2}{2}, -\frac{x_3}{2})^T$, where T denotes the transpose, and $u_{\text{old}} = U + u$

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The Exponentially-Growing Noise

The formula for the solution of the Navier-Stokes equation linearized about the initial flow $U(x)$ is

$$u(x,t) = \sum_{k \neq 0} \int_0^t e^{-(4\nu\pi^2|k|^2 + 2\pi i U_0 k_1)(t-s)} \times$$

$$\begin{pmatrix} e^{-U'(t-s)} & 0 & 0 \\ 0 & e^{\frac{U'}{2}(t-s)} & 0 \\ 0 & 0 & e^{\frac{U'}{2}(t-s)} \end{pmatrix} c_k^{1/2} d\beta_t^k e_k + O(|U'|)$$

if $|U'| << 1$ is small. This is clearly noise that is growing *exponentially* in time, in the x_1 direction, if $U' < 0$

Saturation by the Nonlinearities

- The exponential growth is saturated by the nonlinear terms in the Navier-Stokes equation
- Thus for fully developed turbulence we get the Stochastic Navier-Stokes driven by the large noise

$$\begin{aligned} du &= (\nu\Delta u - u\cdot\nabla u + \nabla\{\Delta^{-1}[trace(\nabla u)^2]\})dt \\ &+ \sum_{k\neq 0} h_k^{1/2} d\beta_t^k e_k \end{aligned} \tag{2}$$

- Determining how fast the coefficients $h_k^{1/2}$ decay as $k \to \infty$ is now a part of the problem

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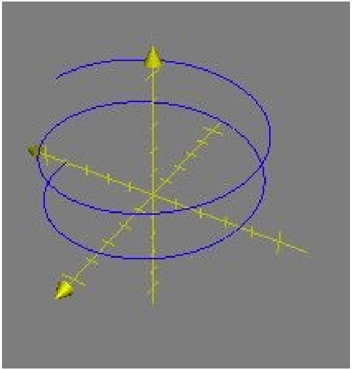
Swirl = Uniform flow + Rotation

- We also need oscillations when the Fourier coefficients $\widehat{u}(0, k_2, k_3, t)$ do not depend on k_1
- To deal with these component we have to start with some rotation with amplitude A, angular velocity Ω and axis of rotation x_1



In which Sobolev Space do we need to work?

- This is the same questions as
  - How rough can the noise be?
  - What Sobolev space dominates $|\nabla u|_4$?
- Sobolev's inequality

$$|\nabla u|_4 \leq C\|u\|_{\frac{n}{2}-\frac{n}{4}+1}$$

- Thus we should work in the Sobolev space $W^{(11/6,2)}$, containing Hölder continuous functions with index $\frac{1}{3}$

The Kolmogorov-Obukhov Scaling

In 1941, Kolmogorov formulated his famous scaling theory of the inertial range in turbulence,

$$S_2(x) = \langle |u(y+x) - u(y)|^2 \rangle \sim (\epsilon |x|)^{2/3},$$

where $y, y+x$ are points in a turbulent flow field, u is the component of the velocity in the direction of x, ϵ is the mean rate of energy dissipation. A Fourier transform yields the Kolmogorov-Obukhov power spectrum in the inertial range

$$E(k) = C\epsilon^{2/3} k^{-5/3},$$

where C is a constant, and k is the wave number.

The Oscillations Induced by the Swirl Make the Existence Theory Work

- Now we find a solution $u + \mathbf{U}$ of the Navier-Stokes equation, with swirl $\mathbf{U} = (U, -A\sin(\Omega t), A\cos(\Omega t))$

As a first guess we use the solution of the linearized equation

$$u_o(x,t) = \sum_{k \neq 0} h_k^{1/2} \int_0^t e^{-(4\pi^2\nu|k|^2 + 2\pi i[Uk_1 + A(k_2,k_3)])(t-s)} d\beta_s^k e_k(x)$$

where $U\, k_1$ and $A(k_2,k_3)/\Omega = A\sqrt{k_2^2 + k_3^2}$, determine the swirl of $\widehat{u}(k,t)$ in fully-developed turbulence, and solve

$$u(x,t) = u_o(x,t) + \int_{t_o}^t K(t-s) * [-u \cdot \nabla u + \nabla \Delta^{-1}(trace(\nabla u)^2)] ds \tag{3}$$

Onsager's Conjecture Dimensions

$\mathcal{L}^2_{(m,p)}$ denotes functions $W^{(m,p)}$, Sobolev norm in $L^2(\Omega, P)$

Theorem

If

$$E(\|u_o\|^2_{(\frac{11}{6}^+,\,2)}) \leq \frac{1}{2}\sum_{k\neq 0}\frac{(1+(2\pi|k|)^{(11/3)^+})}{(2\pi|k|)^2}h_k < \frac{CB^2}{24} \quad (4)$$

where the uniform flow U and the amplitude A are sufficiently large, and Ω *sufficiently small, so that*

$$\text{ess sup}_{t\in[0,\infty)} E(\|u\|^2_{(\frac{11}{6}^+,\,2)})(t) < CB^2 \quad (5)$$

holds, then the integral equation (3) has unique global solution $u(x,t)$ *in the space* $C([0,\infty);\mathcal{L}^2_{(\frac{11}{6}^+,\,2)})$

What are the properties of these solutions?

- These stochastic processes are not smooth in space, they are Hölder continuous with exponent $\frac{1}{3}$
- There is no *blow-up* in finite time, instead the solutions roughens
- The solutions start smooth, $u(x, 0) = 0$, but as the noise gets amplified they roughen, until they have reached the characteristic roughness $\chi = \frac{1}{3}$ in the statistically stationary state
- Neither ∇u nor $\nabla \times u$ are continuous in general

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A Unique Invariant Measure

- We must prove the existence of a unique invariant measure to prove the existence of Kolmogorov's statistically invariant stationary state
- We define the invariant measure by the limit

$$\lim_{t \to \infty} E(\phi(u(t))) = \int_H \phi(u) d\mu(u) \tag{6}$$

Existence and Uniqueness of the Invariant Measure

- This limit exist if we can show that the sequence of associated probability measures are tight
- If the N-S semigroup maps bounded function on H onto continuous functions on H, then it is called Strongly Feller
- Irreducibility says that for an arbitrary b in a bounded set

$$P(sup_{\{t<T\}}\|u(t)-b\| < \epsilon) > 0$$

Outline

Kolmogorov's Conjecture

- We want to prove Kolmogorov's statistical theory of turbulence
- Kolmogorov's statistically stationary state (3-d)

$$S_2(x,t) = \int_H |u(x+y,t) - u(y,t)|_2^2 d\mu(u) \sim |x|^{2/3}$$

Theorem

In three dimensions there exists a statistically stationary state, characterized by a unique invariant measure, and possessing the Kolmogorov scaling of the structure functions

$$S_2(x) \sim |x|^{2/3}$$

Proof of Kolmogorov's Conjecture

Proof.

For x small the statement follows by Hölder continuity. But the scaling of $S_2(x)$ can be directly estimated. Thus

$$S_2(x) \leq C|x \cdot (L - x)|^{2/3}$$

where C is a constants, $x \in \mathbb{T}^3$, L is a three vector with entries the sizes of the faces of $\mathbb{T}^3$ and $|x| \leq |L|$. □

Corollary

The solution of the Stochastic Navier-Stokes equation is not in H^{2^-}

The Approximate Invariant Measure

The invariant measure of (2) can actually be computed explicitly

$$d\mu = M_\infty d\,[\mathcal{N}(0,\sqrt{2\nu}) * \mathcal{N}(0,Q_\infty)] \qquad (7)$$

where the variance Q_∞ is

$$Q_\infty^{-1} = \sum_{k\neq 0} \frac{h_k}{2\nu\lambda_k}$$

with coefficients $h_k = |h_k^{1/2}|^2$. In terms of densities the invariant measure is

$$d\mu = e^{\{-\int_0^\infty (\mathbf{U}+u(x,s))\cdot dx - \frac{1}{2}\int_0^\infty |\mathbf{U}+u(x,s)|^2 ds\}} \frac{e^{-\frac{|x|^2}{2\nu}}}{\sqrt{2\nu}} dx$$

$$\times \prod_{k\neq 0} \frac{e^{-\frac{h_k \hat{u}_k^2}{2\nu\lambda_k}}}{\sqrt{2\nu\lambda_k/h_k}} d\hat{u}_k$$

Outline

Conclusions

- Starting with sufficiently fast constant flow in some direction there exist turbulent solutions
- These solutions are Hölder continuous with exponent $\frac{1}{3}$ (Onsager's conjecture)
- There exist a unique invariant measure corresponding to these solutions
- This invariant measure give a statistically stationary state where the second structure function of turbulence scales with exponent $\frac{2}{3}$ (Kolmogorov's conjecture)
- In one and two dimensions the same works but the scaling exponents are respectively $3/2$ (Hack's Law) and 2 (Batchelor-Kraichnan Theory)

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Applications

- Better closure approximations for RANS
- Better subgrid models for LES
- Applications to Rayleigh-Bénard experiments (Ahlers, UCSB) and Taylor-Couette (LANL)
- For RB we want to compute the heat transport

The Artist by the Water's Edge
Leonardo da Vinci Observing Turbulence