Elastic theory of unconstrained non-Euclidean plates

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Abstract

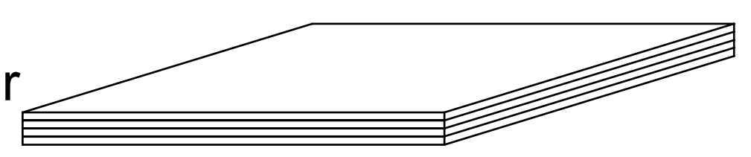
Non-Euclidean plates are thin elastic bodies having no stress-free configuration. Such bodies exhibit residual stress when relaxed from all external constraints and may assume complicated equilibrium shapes even in the absence of external forces. We present a mathematical framework for such bodies in terms of a covariant theory of linear elasticity, valid for large displacements and arbitrary intrinsic geometry.

Plates vs. Shells

Plates are bodies that posses no structure variation across their thickness. A plate may be described as a collection of identical surfaces.

A Plate

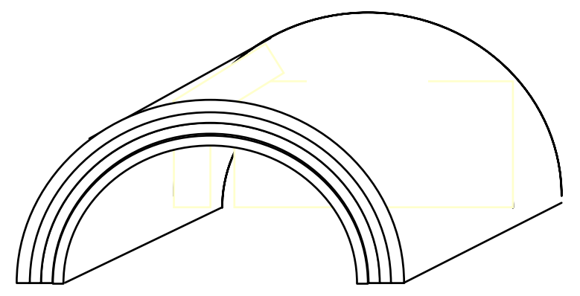
Symmetric under reflection



Shells are bodies that posses structure variation across their thickness. A shell may be described as a collection of non-identical surfaces.

A Shell

Not symmetric under reflection

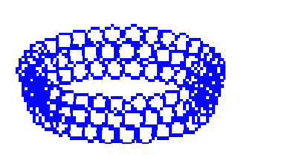


Gaussian curvature must vanish

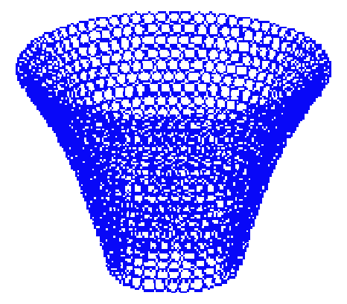
Non-Euclidean plates:

Neither plates nor shells

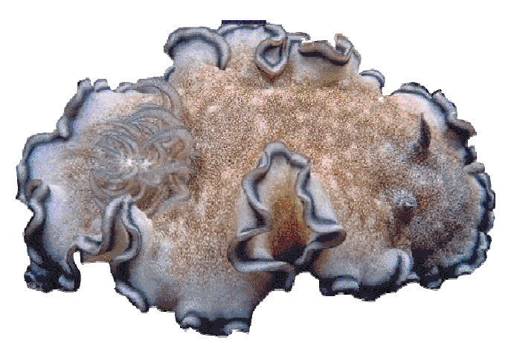
Symmetry under reflection & Non-vanishing Gaussian curvature



Growth toy model: At each generation the upper most ring of N cells creates a new ring of λN cells (1< λ). Such a growth creates a surface of constant negative Gaussian curvature.

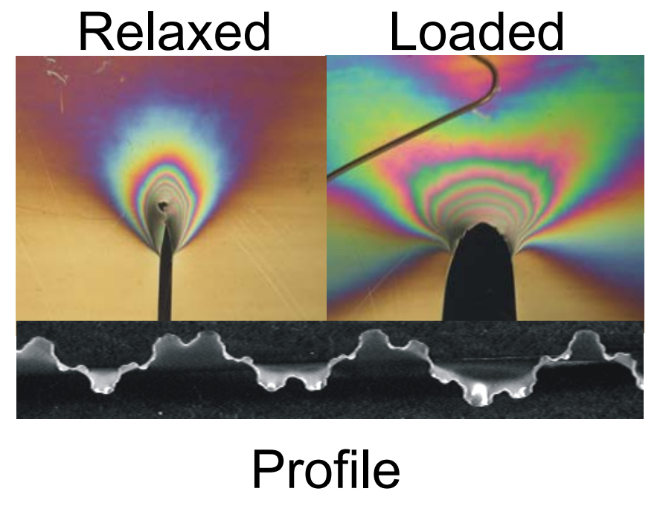


2D biological growth: “Most metrics” are non-flat. An isotropic growth rule which depends on some scalar field, ϕ, (growth hormone, etc.), takes the form of $ds^2 = e^{F(\phi)}(dx^2+dy^2)$.
In order to keep a flat surface flat growth must be very regulated:

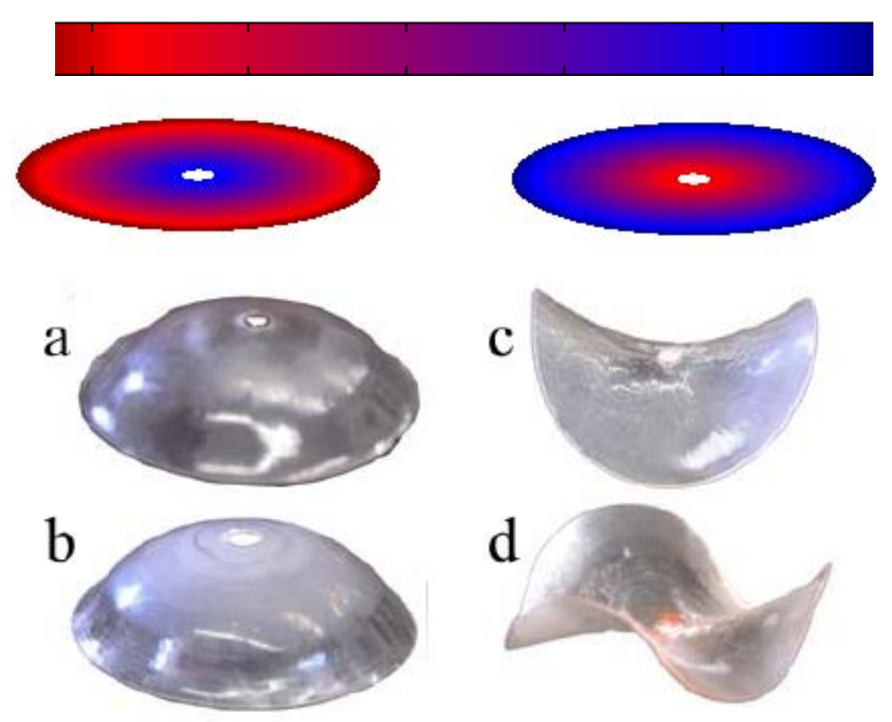


$$K = 0 \Leftrightarrow F'(\phi)(\nabla^2\phi) + F''(\phi)(\nabla\phi)^2 = 0$$

Tearing: Loading a cracked ductile sheet results in stress focusing at the crack tip (shown qualitatively using photo-elasticity). The stress deforms the sheet plastically, resulting in non-uniform lateral lengthening.



Mimicking non homogeneous growth using programmable responsive gels:
- A thermo-responsive hydro-gel undergoes a volume reduction when heated.
- The local volume reduction ratio may be controlled chemically.
- An initially flat disc may be “programmed” to shrink non-homogeneously.



Thickness 0.75mm.

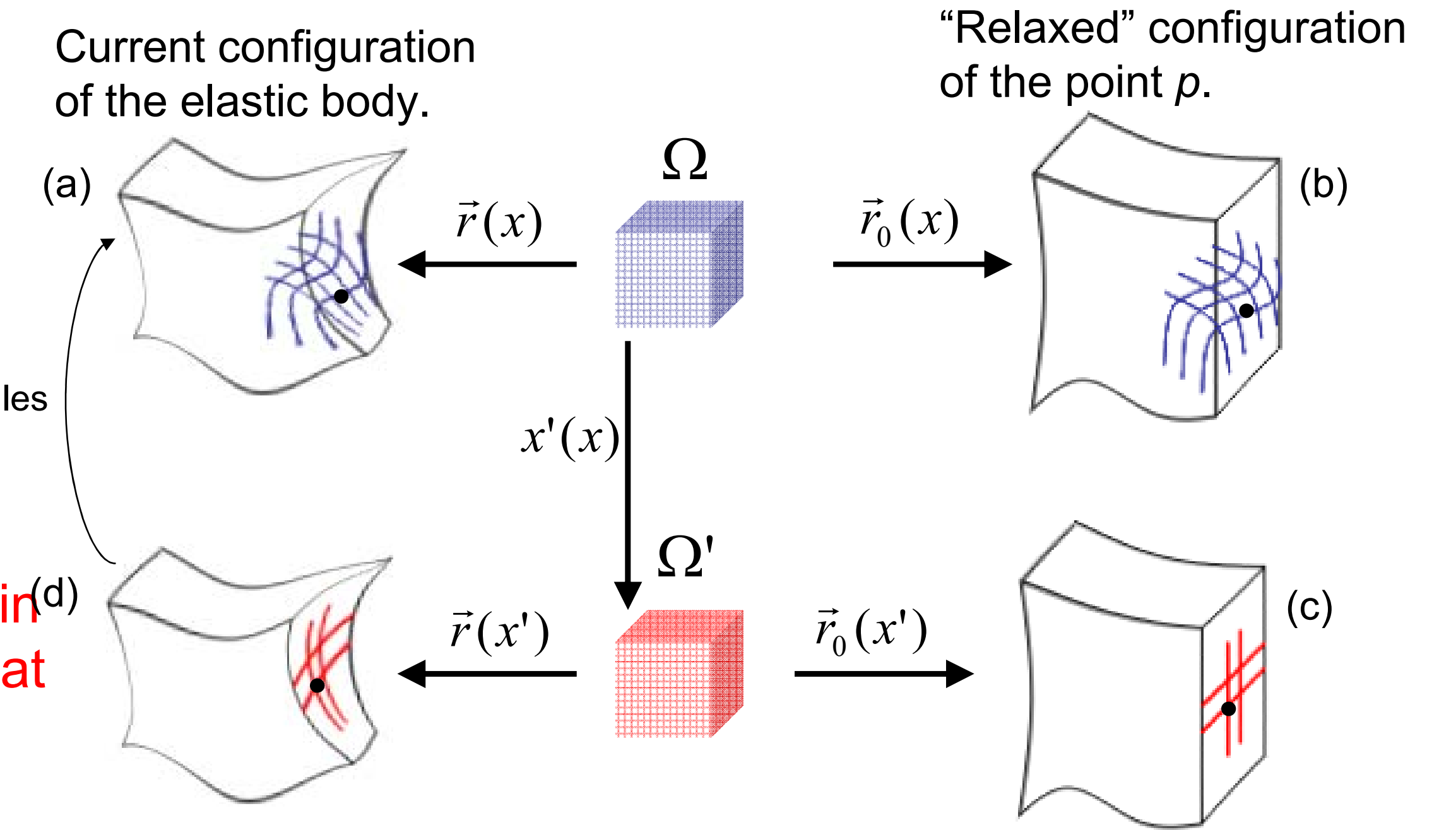
Thickness 0.6mm.

Discs radius 30mm

$K_a = K_b = 0.11\text{cm}^{-2}$ $K_c = K_d = -0.11\text{cm}^{-2}$

“Incompatible” 3D elasticity

- Elastic forces are local and non-dissipative. $E = \int_{\mathcal{D}} \tilde{w}\sqrt{|g|}dx^1dx^2dx^3$
- A 3D metric tensor defines the physical configuration in R^3 uniquely. $\tilde{w}(g,\boldsymbol{x})$
- We assume that the energy density is non-negative, $\tilde{w}(g,\boldsymbol{x}) \geq 0$, and at every point possesses a unique zero: $\tilde{w}(g,\boldsymbol{x}) = 0 \Leftrightarrow g = \bar{g}(\boldsymbol{x})$



Some set of curvilinear coordinates.

A special set of curvilinear coordinates in which the target metric at *p* is the identity tensor.

(d) The strain reads $\varepsilon = \frac{1}{2}(g-\mathbf{I}) = \frac{1}{2}((\boldsymbol{\nabla r})^T\boldsymbol{\nabla r} - \mathbf{I}) = \frac{1}{2}(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T + (\boldsymbol{\nabla u})^T\boldsymbol{\nabla u})$

The energy density is known: $w' = \frac{1}{2}A'^{ijkl}\varepsilon'_{ij}\varepsilon'_{kl}$, where $A'^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$

(a) The pullback transformation gradient $\Lambda = d\boldsymbol{x}/d\boldsymbol{x}'$ is used to return to the original set of curvilinear coordinates $g_{ij} - \bar{g}_{ij} = 2\varepsilon_{ij} = 2\Lambda_i^k\Lambda_j^l\varepsilon'_{kl} = \Lambda_i^k\Lambda_j^l(g'_{kl} - \delta_{kl}) = g_{ij} - \Lambda_i^k\Lambda_j^l\delta_{kl}$

The Target metric, $\Lambda_i^k\Lambda_j^l\delta_{kl} = \bar{g}_{ij}$, must be symmetric, and positive definite (a metric).
The target metric does not need to satisfy the compatibility conditions.

Total elastic energy

$$E(g) = \int_\Omega w(g)\sqrt{|\bar{g}|}dx^1dx^2dx^3$$

Energy density per unit volume of the target metric

$$w = \tfrac{1}{2}A^{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

The elasticity tensor $A^{ijkl} = \lambda\bar{g}^{ij}\bar{g}^{kl} + \mu\left(\bar{g}^{ik}\bar{g}^{jl} + \bar{g}^{il}\bar{g}^{jk}\right)$

Green St. Venant strain tensor

$$\varepsilon_{ij} = \tfrac{1}{2}(g_{ij} - \bar{g}_{ij})$$

Second Piola Kirchhoff stress

$$S^{ij} = A^{ijkl}\varepsilon_{kl}$$

Equilibrium equations:

$$\bar{\nabla}_j S^{ij} + (\Gamma^i_{jk} - \bar{\Gamma}^i_{jk})S^{jk} = 0 \qquad \text{in } \mathcal{D}$$
$$S^{ij}n_j = 0 \qquad \text{on } \partial\mathcal{D},$$

Greek indices *i,j,k…* assume the values {1,2,3}

Non-Euclidean plates

A plate is a continuous collection of identical surfaces

A plate allows a parameterization in which the 3D target metric takes the following form

$$\bar{g}_{ij} = \begin{pmatrix} \bar{g}_{11} & \bar{g}_{21} & 0 \\ \bar{g}_{12} & \bar{g}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \partial_3\bar{g}_{ij} = 0.$$

A plate is called even if the domain $\mathcal{D} \subset \mathbb{R}^3$ of the curvilinear coordinates may be decomposed into $\mathcal{D} = \mathcal{S} \times [-\frac{h}{2}, \frac{h}{2}]$ $\mathcal{S} \subset \mathbb{R}^2$

A plate is called thin if its thickness is much smaller than the other lateral dimensions.

We consider the mid-surface (x^3=0) as a 2D surface $\boldsymbol{R}(x^1,x^2) = \boldsymbol{r}(x^1,x^2,0)$

The first fundamental form (metric) of the mid-surface $a_{\alpha\beta} = \partial_\alpha\boldsymbol{R}\cdot\partial_\beta\boldsymbol{R}$

The second fundamental form of the mid-surface $b_{\alpha\beta} = \partial_\alpha\partial_\beta\boldsymbol{R}\cdot\hat{\mathbf{N}}$

Elastic theory of thin Non-Euclidean plates

For thin plates, with the aid of the Kirchhoff Love assumptions it is possible to formulate a reduced elastic energy in terms of 2D surface properties.

$$E = hE_S + h^3E_B$$

$E_S = \frac{1}{2}\int_S \mathcal{A}^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\sqrt{|\bar{g}|}\,dx^1dx^2$ Stretching content

$E_B = \frac{1}{24}\int_S \mathcal{A}^{\alpha\beta\gamma\delta}b_{\alpha\beta}b_{\gamma\delta}\sqrt{|\bar{g}|}\,dx^1dx^2$ Bending content

$\varepsilon_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} - \bar{g}_{\alpha\beta})$ The 2D strain is estimated at the mid-surface

$$\mathcal{A}^{\alpha\beta\gamma\delta} = \frac{Y}{1+\nu}\left(\frac{\nu}{1-\nu}\bar{g}^{\alpha\beta}\bar{g}^{\gamma\delta} + \bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta}\right)$$

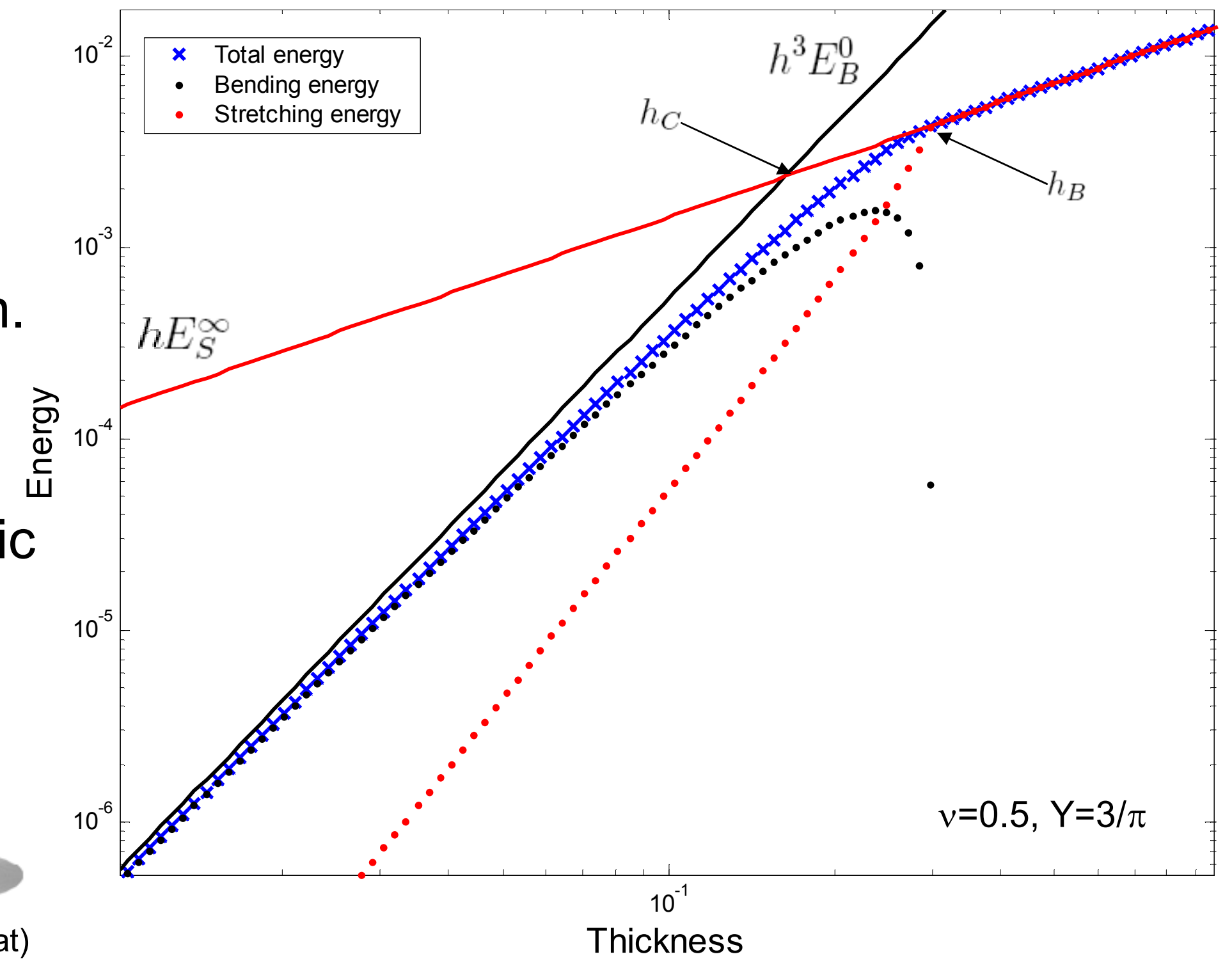
Latin indices α,β,γ,δ… assume the values {1,2}

Numerical results

We minimize the reduced 2D plate energy with respect to a target metric of a punctured hemisphere: $\bar{g} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(r) \end{pmatrix}$

The solution is assumed to be axially symmetric. $0.1 \leq r \leq 1.1$

- For large thickness the plate remains unbuckled and its energy is linear in the thickness. The prefactor E_S^∞ is the stretching content of the plane stress solution.
- For small thickness the plate approaches the isometric configuration and its energy is cubic in the thickness. The prefactor E_B^0 is the bending content of the isometric solution.

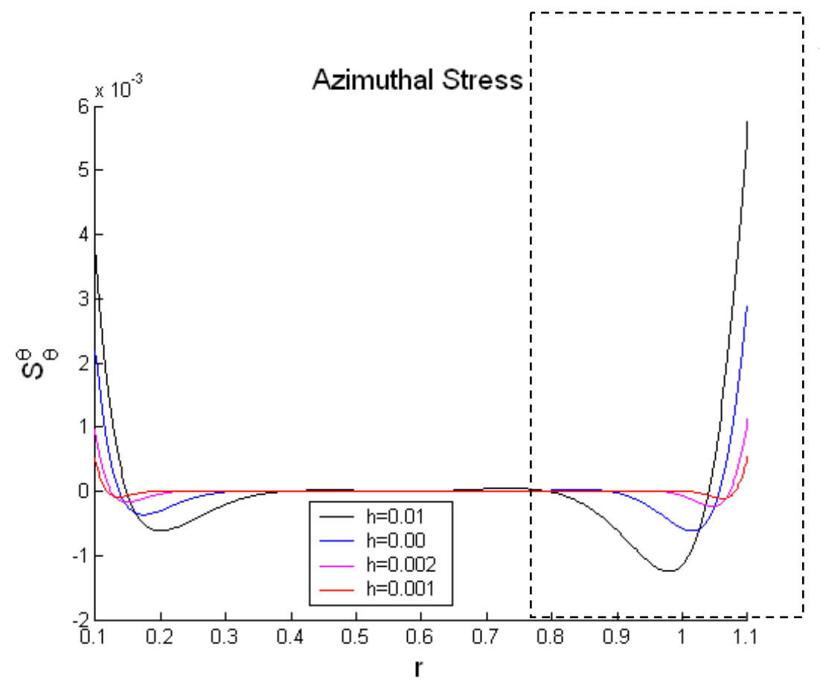


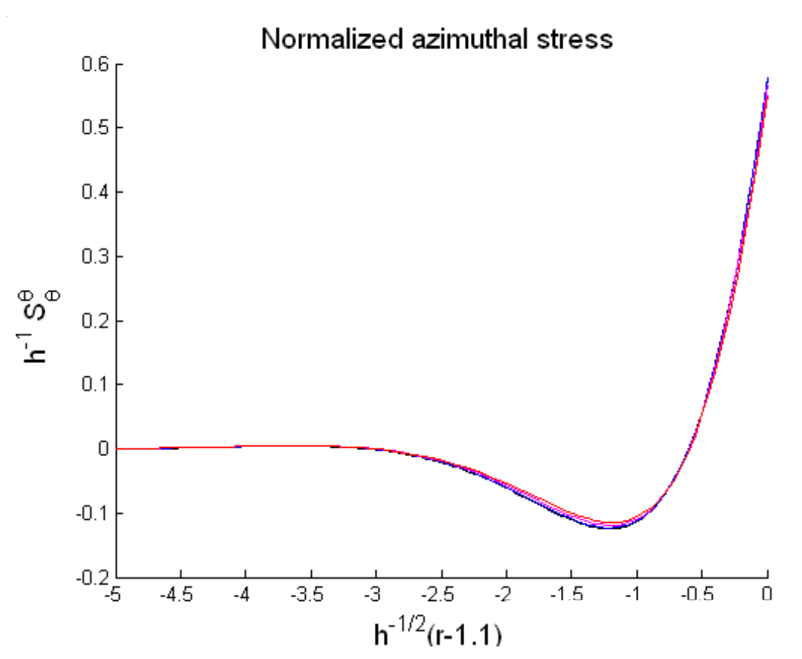
Almost isometric h = 0.01

Slightly bent h = 0.2

Unbuckled (flat) h = 0.4

- Aside from a small crossover region, the bending and stretching energy terms are incomparable.
- The crossover thickness $h_C = \sqrt{E_S^\infty/E_B^0}$ plays an important role in the post buckling behavior.





- At the free edges a boundary layer of thickness $h^{1/2}$ appears.
- The different components of the stress scale with different powers of h, yet exhibit the same boundary layer thickness.

- For the case described above, buckling occurs through a supercritical bifurcation.
- For different settings (plate metric), subcritical bifurcations have been observed.