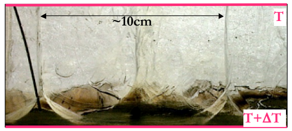
Viscous entrainment of thin tendrils from stratified liquid layers

Laura E. Schmidt and Wendy W. Zhang

Department of Physics and the James Franck Institute, University of Chicago

Motivation & Context

Experiments: 2-layer thermal convection of miscible liquids



Jellinek & Manga, Reviews of Geophysics, **42** (2004).

Localized tendrils persist & stabilize overall flow.

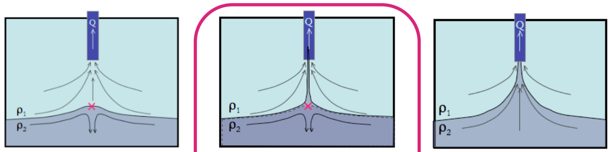
Mixing by flow through entrained tendrils.

tendril radius << imposed length scales

Experimental studies: Davaille, *JFM* (1999); Le Bars and Davaille, *JFM* (2004) Jellinek and Manga, *Nature* 2002

ENTRAINMENT REGIMES

withdrawal flux, Q, increasing →



hump — no entrainment; layers recirculate; stagnation point at tip

thin tendril — liquid at surface entrained; stagnation point at base

thick spout — bulk upwards flow

thick spout state studied theoretically (Sleep, *Geophys. J.* 1988)

Recent experiments show existence of **stagnation point** for **thin tendrils** (Davaille, 2006; Case and Nagel, *PRL* 2007).

Need a simple understanding:

What sets the size of the tendril?

What is the flow through the tendril?

Entrainment in a Model Flow

Drive entrainment with straining flow in upper layer, consistent w/ stagnation pt $\mathbf{U}=(-Er/2, 0, Ez)$, where E is the strain rate (s^{-1})

Flow induced in lower layer via viscous coupling at interface.

Steady-state flow dynamics separate into **two distinct regions**:

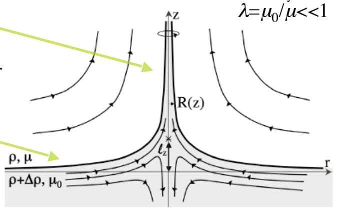
viscosity ratio $\lambda=\mu_0/\mu<<1$

1) Near tendril

Long & slender interface.
Ext. viscous stress ~ interior pressure.

2) Far field, away from tendril

Interface almost flat.
Ext. viscous stress ~ stratification.



Simple Scaling Arguments

Natural vertical scale: balance between viscous stress & stratification:

$$\ell_z = 2\mu E/\Delta\rho g$$

To find radial length scale, look at vertical flow inside tendril. 3 contributions:

1) upwards plug flow, driven by exterior $u_+(z) = Ez$

2) Poiseuille flow from vertical pressure gradient $u_P(r,z) = -\frac{dP_0/dz}{4\mu_0}(R^2(z)-r^2)$

3) drainage due to hydrostatic pressure gradient $u_-(r,z) = -\frac{\Delta\rho g}{4\mu_0}(R^2(z)-r^2)$

Interior pressure, P_0 must balance inward press from exterior viscous stress, in limit $\lambda=\mu_0/\mu<<1$: $P_0 = O(\mu E)$

For flows to be of same size, and yield stagnation point, requires relation:

$$\ell_R = \sqrt{2\lambda}\ell_z$$

since λ is small, consistent w/ slender shape

Estimate volume flux Q_0 through tendril, $Q_0 \sim U_z$ (Area)

$$Q_0 = c_0(E\ell_z)(\pi\ell_R^2) = c_0(16\pi\mu^2\mu_0E^4)/(\Delta\rho g)^3$$

$O(1)$ constant to be determined

Note: similar arguments can be made for entrainment of 2D sheets

Long-Wavelength Model for Tendril

A more precise analysis of the tendril dynamics yields

$$Q_0 = (Ez)\pi R^2(z) - \frac{\pi R^4(z)}{8\mu_0}\left(\frac{dP_0}{dz}+\Delta\rho g\right). \qquad P_0 = 2\mu E\left(1+\frac{z}{R(z)}\frac{dR}{dz}\right).$$

comprising a 2nd order non-linear ODE for radius $R(z)$.

Asymptotic limits are: $R_\infty(z) = \sqrt{Q_0(c_0)/(\pi Ez)}$ as $z\to\infty$

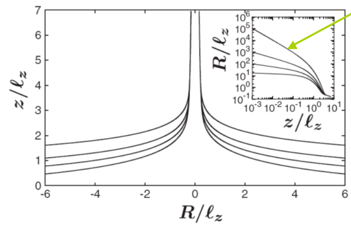
$$R_s(z) = B\left(\frac{\ell_z}{z}\right)^\alpha e^{-z/\ell_z} \quad \text{as } z\to 0$$

allows any power law form with exponent α, and coefficient B, characterizing the width and slope of the tendril near the base.

Continuous Family of Steady-State Shapes

Check above asymptotics by numerical integration of ODE for $R(z)$ for fixed Q_0.

Find continuous family of solutions, with different upstream ($z\to 0$) power laws.



degenerate

Must use another piece of information – the exact large-scale interface deflection – to find a unique solution.

Shape of Deflected Interface

Make specific choice for interface deflection by placing point force, F, at a height S above origin. Straining flow generated, as required, with $E=F/2\pi\mu S^2$.

Solve for deflection by General Reciprocal Theorem (~method of images, Lee et al *JFM* '79) & normal stress balance (**visc. stress ~ gravity**)

$$R_I(z) = S\sqrt{(3\ell_z/2z)^{2/5}-1}$$

Exactly how the interface levels out depends on the geometry of the driving global flow (could use point sink, vortex ring, etc.)

The physical tendril solution is the one which **smoothly** joins the interface.

Only One Tendril Smoothly Joins Interface

Vary c_0 (vary Q_0) and find only one tendril solution can smoothly join $R_I(z)$

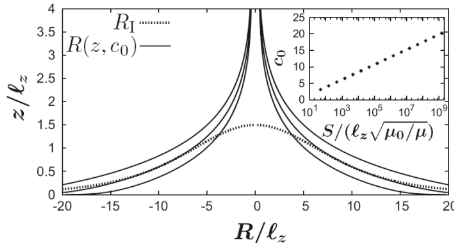


FIG. 4. Rescaled tendril solutions $R(z)$ for $c_0 = 3.4$, 3.1, and 2.9 (solid lines, top to bottom), together with the large-scale deflection solution $R_I(z)$ (dotted line) for $S/\ell_z = 15$ and $\mu_0/\mu = 0.1$. Only $c_0 = 3.1$ allows the two shapes to merge smoothly. Inset shows c_0 increases logarithmically with $S/(\ell_z\sqrt{\mu_0/\mu})$.

However, details of geometry of global flow (here, point force) have very weak, logarithmic effect on volume flux.

$$Q_0 = \frac{16\pi\mu^2\mu_0E^4}{(\Delta\rho g)^3}\left[\gamma_1\log\left(\frac{S}{\ell_z\sqrt{\mu_0/\mu}}\right)+\gamma_2\right],$$

γ_1, γ_2 depend on choice of driving flow (here point force)

Conclusions

Tendril width is selected by competition between viscous stress and gravity

Volume flux is strongly dependent on E and linear with λ. These trends should be possible to see in experiments.

Volume flux is very weakly dependent on global details (S, $R_I(z)$), so expect entrainment in experiments to be insensitive to large-scale fluctuations.