Blowup dynamics of an unstable thin-film equation

IMA Summer Program

Geometrical Singularities and Singular Geometries

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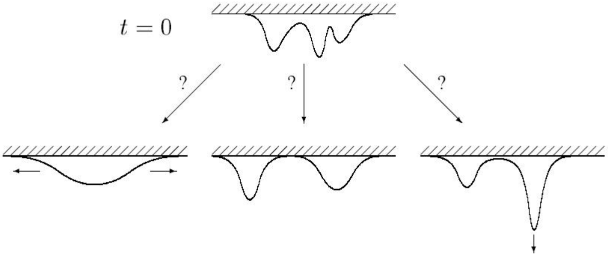
Thin-film equations

(TFE) $$u_t = -(u^n u_{xxx})_x - (u^m u_x)_x \qquad \text{on } \mathbb{R} \times [0, T]$$



- Describe the evolution of a thin layer of fluid under the effects of destabilizing forces, like gravity.
- Zero contact angle
- Energy

$$E(u) = \int \frac{1}{2} |\nabla u|^2 - \frac{1}{(m-n+2)(m-n+1)} u^{m-n+2}.$$

Gradient flow structure

Equation

$$u_t = -\nabla \cdot \left(u^n \nabla \left(\Delta u + \frac{u^{m-n+1}}{m-n+1} \right) \right)$$

Metric — set by the dissipation mechanism

Let s_1, s_2 be tangent vectors at u (zero-mean functions)

$$\langle s_1, s_2 \rangle_u = \int u^n \nabla p_1 \cdot \nabla p_2$$

where $\quad -\nabla \cdot (u^n \nabla p_i) = s_i \quad$ for $i = 1, 2$.

Gradient flow

$$\langle u_t, s \rangle_\rho = -\frac{\delta E}{\delta \rho}[s]$$

for all tangent vectors s.

Gradient flow structure

Equation

$$u_t = -\nabla \cdot \left(u^n \nabla \left(\Delta u + \frac{u^{m-n+1}}{m-n+1} \right) \right)$$

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Gradient flow

$$\langle u_t, s \rangle_\rho = -\frac{\delta E}{\delta \rho}[s] = \int \left(\Delta u + \frac{u^{m-n+1}}{m-n+1} \right) s \, dx$$

for all tangent vectors s.

Gradient flow structure

Equation

$$u_t = -\nabla \cdot \left(u^n \nabla \left(\Delta u + \frac{u^{m-n+1}}{m-n+1} \right) \right) = \nabla \cdot \left(u^n \nabla \left(\frac{\delta E}{\delta u} \right) \right)$$

Metric — set by the dissipation mechanism

Let s_1, s_2 be tangent vectors at u (zero-mean functions)

$$\langle s_1, s_2 \rangle_u = \int u^n \nabla p_1 \cdot \nabla p_2 = \int p_1 s_2$$

where $\quad -\nabla \cdot (u^n \nabla p_i) = s_i \quad$ for $i = 1, 2$.

Gradient flow

$$\langle u_t, s \rangle_\rho = -\frac{\delta E}{\delta \rho}[s] = \int \left(\Delta u + \frac{u^{m-n+1}}{m-n+1} \right) s \, dx$$

for all tangent vectors s.

Gradient flows wrt Wasserstein metric

Equation: $$u_t = -(u\,u_{xxx})_x - (u^m u_x)_x$$

Configuration space: $$\mathcal{M} = \left\{ u \geq 0, \; : \; \int u = M > 0, \; \int |x|^2 u dx < \infty \right\}$$

Energy: $$E(u) = \int \frac{1}{2} u_x^2 - \frac{1}{m(m+1)} u^{m+1} dx$$

Metric: $$\langle s_1, s_2 \rangle_u = \int u \nabla p_1 \cdot \nabla p_2$$

where $$-\nabla \cdot (u \nabla p_i) = s_i \qquad \text{for } i = 1, 2.$$

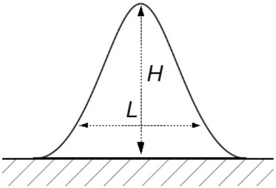
- s_i – Eulerian tangent vector, ∇p_i – Lagrangian
- $(\mathcal{M}, \langle \cdot, \cdot \rangle_u)$ is a manifold.
- The induced distance is the Wasserstein distance:

$$d(u_1, u_2)^2 = \inf_{\Phi_\# u_1 = u_2} \int |\Phi(x) - x|^2 u_1(x) dx$$

Dynamics of TFE

$$u_t = -(u^n u_{xxx})_x - (u^m u_x)_x$$

Q: When are the stabilizing and destabilizing forces in balance?



$$(u^n u_{xxx})_x \sim \frac{H^{n+1}}{L^4} = H^{n+5}$$

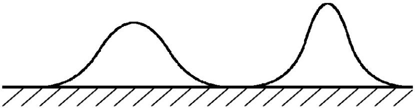
$$(u^m u_x)_x \sim \frac{H^{m+1}}{L^2} = H^{m+3}$$

in balance if $\boxed{m=n+2}$

- If $m < n+2$ Bertozzi and Pugh have shown that weak solutions exist for all time.
- Conjecture: Blowup is possible if $m \geq n+2$. (Has been proven for $n = 1$ by Bertozzi and Pugh)
- If $m = n+2$ selfsimilar blowup solutions exist when $0 < n < 3/2$ (Pugh and S.)

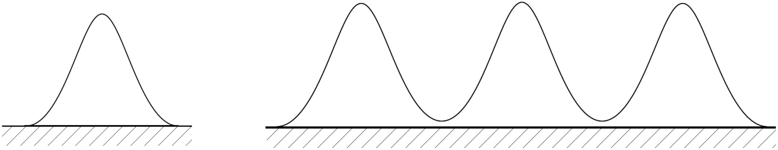
Selfsimilar solutions

- **Steady states** can have any mass if $m \neq n+2$. When $m = n+2$ all droplet steady states have the same mass, M_c, and the family of steady states is dilation invariant.



- **Source-type (spreading) selfsimilar solutions** exist when $0 < n < 3$ (Beretta).
- **Blow-up selfsimilar solutions** exist when $0 < n < 3/2$ (Pugh and S.)

$$u(x,t) = (1-t)^{-1/(n+4)}\,\rho(x\,(1-t)^{-1/(n+4)})$$



Stability of steady states

Linearizing a gradient flow

Linearized operator at a steady state is symmetric in the metric of the flow

- Linearized operator given by the Hessian of E at η.
- The construction of the inner product on $\mathcal{M}$ suggests the use of particular coordinates on $T\mathcal{M}$.

$$s \longrightarrow f \qquad \text{where} \quad -(\eta f)_x = s$$

Metric is weighted L^2 inner product $\langle f_1, f_2 \rangle = \int \eta f_1 f_2$.

- $$H(f) := \frac{HessE(f,f)}{\langle f, f \rangle_\eta} = \frac{E(\gamma(t))''}{\langle f, f \rangle} = \frac{\displaystyle\int \eta^2 f_{xx}^2 - \frac{m-3}{m+1}\, \eta^{m+1}\, f_x^2 \, dx}{\displaystyle\int \eta f^2}$$
- $f = 1$ corresponds to translations; $H(1) = 0$
- $f = x$ corresponds to dilations
  - If $m > 3$ then $H(x) < 0$ – an unstable direction
  - If $m = 3$ then $H(x) = 0$ – a neutral direction
  - If $m < 3$ then $H(x) > 0$ and moreover $H(f) > \lambda > 0$ for all f such that $\langle f, 1 \rangle = 0$.

Stability of selfsimilar solutions when $n = 1$, $m = 3$

- All droplet steady states have the same mass, denote it M_c.
- Initial data with mass less than M_c do not blow up.
- Spreading selfsimilar solutions are linearly stable.

Stability of blowup profiles ρ

- In similarity variables the equation becomes:

$$w_t = -\left(ww_{xxx} + w^3 w_x + \frac{xw}{5}\right)_x.$$

- It is a gradient flow of energy

$$\tilde{E} = \int \frac{1}{2} w_x^2 - \frac{1}{12} w^4 - \frac{x^2 w}{10}\, dx.$$

- The quadratic form is:

$$H(f) = \frac{\int \rho^2 f_{xx}^2 - \frac{4}{5}\int_{|x|}^{L} s\rho(s)ds - \frac{1}{5}\rho f^2 dx}{\int \rho f^2}$$

Stability of selfsimilar blowup solutions

Equation: $$w_t = -\left(ww_{xxx} + w^3 w_x + \frac{xw}{5}\right)_x$$

Energy: $$\tilde{E} = \int \frac{1}{2} w_x^2 - \frac{1}{12} w^4 - \frac{x^2 w}{10}\, dx$$

quadratic form: $$H(f) = \frac{\displaystyle\int \rho^2 f_{xx}^2 - \frac{4}{5}\int_{|x|}^{L} s\rho(s)ds - \frac{1}{5}\rho f^2 dx}{\displaystyle\int \rho f^2}$$

- Note that $H(1) < 0$ and $H(x) < 0$.
- For single-bump profiles ρ that have been constructed $H(f) > \lambda > 0$ for all f such that $\langle f, 1\rangle = 0$ and $\langle f, x\rangle = 0$.
- For multi-bump profiles there exist other unstable directions.

Unstable Thin Film

$$u_t = -(u^n u_{xxx})_x - (u^m u_x)_x$$

Nonlinear Schrödinger

$$i\psi_t + \Delta\psi + |\psi|^{2\sigma} = 0$$

critical powers

$$m = n + 2 \qquad\qquad \sigma = \frac{2}{d}$$

conserved quantity:

L^1-norm of u $\qquad$ L^2-norm of $|\psi|$

dissipates the energy:

$$E = \int \frac{1}{2} u_x^2 - c u^{m-n+2} dx$$

conserves the Hamiltonian:

$$H = \int |\nabla\psi|^2 - \frac{1}{\sigma+1} |\psi|^{2\sigma+2} dx$$

Gagliardo–Nirenberg inequality

$$\int f^4 dx \leq 6 \frac{\|f\|_{L^1}^2}{\|\eta\|_{L^1}^2} \|f_x\|_{L^2}^2 \qquad\qquad \int f^{2\sigma+2} \leq \frac{d+2}{d} \frac{\|f\|_{L^2}^{2\sigma}}{\|R\|_{L^2}^{2\sigma}} \|\nabla f\|_{L^2}^2$$

η droplet steady state $\qquad$ R ground state *with P. Raphael:*

Rate of the blowup

- **Theorem** If u blows up at time T then

$$\|u_x(\cdot,t)\|_{L^2} \geq (T-t)^{-3/10}$$

- Key: Lower bound on the time of existence of solutions that depends only on $\|u_x\|_{L^2}$ and $\|u\|_{L^1}$ (Bertozzi and Pugh).

Mass concentration

- **Theorem** Let u be a solution that blows up at time T and $t_n \to T$. If

$$u(\cdot,t_n) \rightharpoonup f dx + \sigma \quad \text{with } \sigma \perp dx$$

then $\|u\|_{L^1} - \|f\|_{L^1} \geq \|\eta\|_{L^1}$. Here η is the droplet steady state.

Open problems

On dynamics of TFE

- Linear stability of selfsimilar solutions when $n \neq 1$
- Asymptotic (nonlinear) stability
- Establish blowup for a large class of initial data when $m \geq n + 2$ (known when $n = 1$)
- Show that blowup is generic when in the critical case, $m = n + 2$, when the mass is greater then M_c
- Asymptotic shape and rate when $n \geq 3/2$ and $m = n + 2$