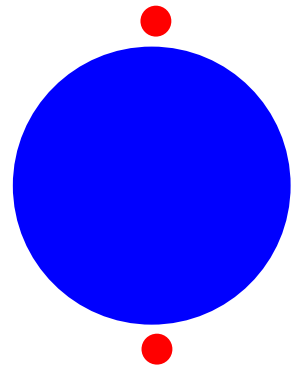
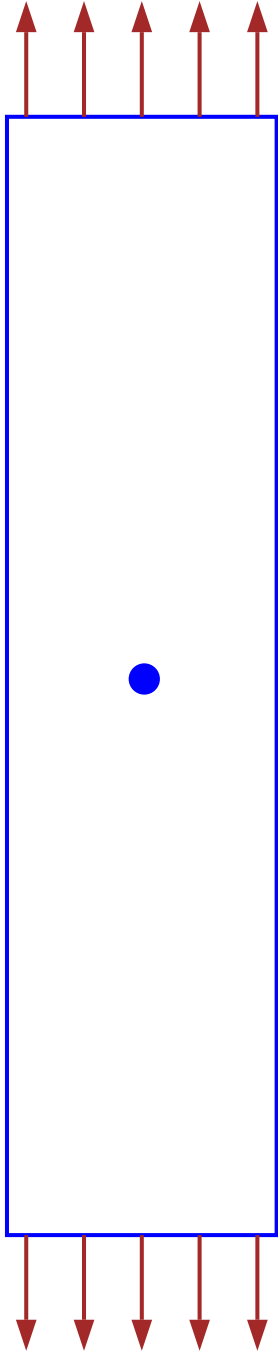
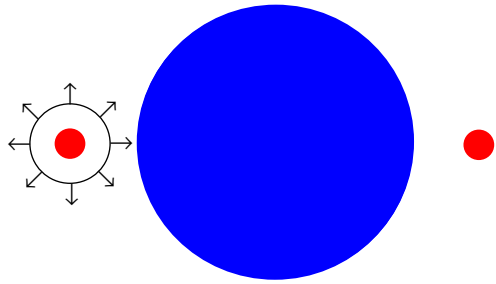


On the Symmetry of Singular Minimizers

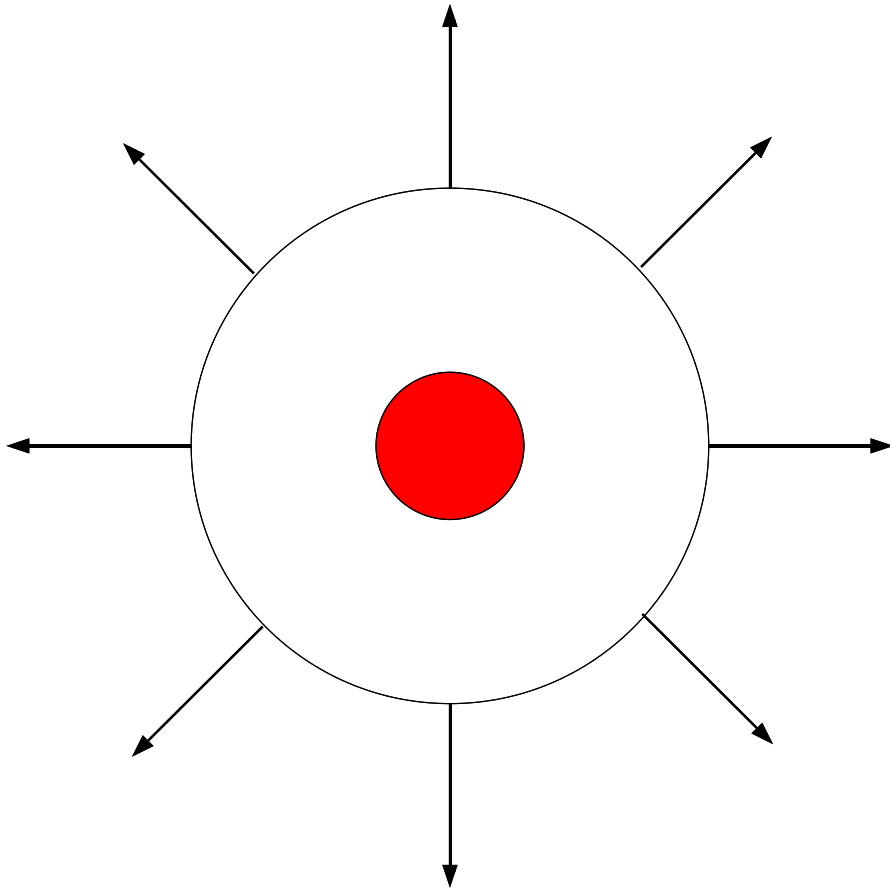
in Nonlinear Elasticity

[Joint with J. Sivaloganathan (Bath)]

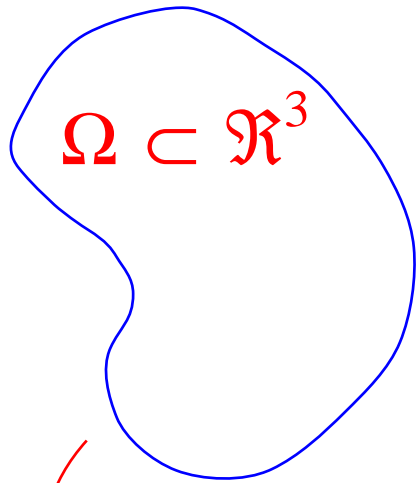




Assume that the loading
at the cavitation point is
equal triaxial tension.



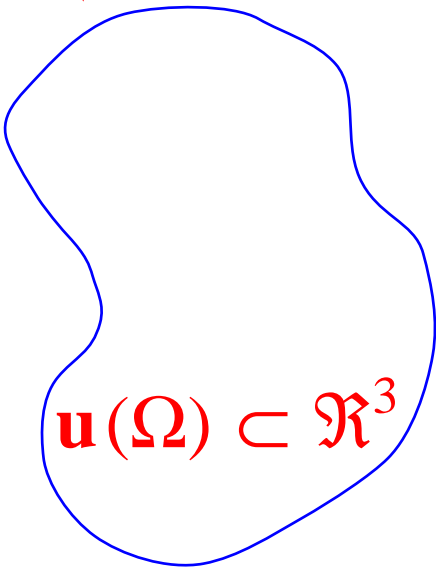
Is the shape of the hole
(asymptotically) spherical?



$\mathbf{u} : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ – Deformation

$F_{ij} := (\nabla \mathbf{u})_{ij} = \partial u_i / \partial x_j$ – Deformation Gradient

\mathbf{u}



$W(\mathbf{F})$ – Stored Energy

$\det \mathbf{F}(\mathbf{x}) \equiv 1$ – Incompressible

$$W(\mathbf{F}) = \Psi(|\mathbf{F}|^p, |\operatorname{adj} \mathbf{F}|^q, \det \mathbf{F}),$$

$\Psi \geq 0$... convex, monotone increasing in each of its first 2 arguments

Cavitation Requires: $1 \leq p < 3, 1 \leq q < 3/2$

In particular, the stored energy is homogeneous, isotropic, and polyconvex.

$$\min_{\substack{\mathbf{u}(\mathbf{x})=\mathbf{d}(\mathbf{x}) \\ \mathbf{x} \in \partial\Omega}} E(\mathbf{u}) := \int_{\Omega} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}$$

Deformations = $\{ \mathbf{u} \in W^{1,1}(\Omega; \mathfrak{R}^3) : \det \nabla \mathbf{u} > 0 \text{ a.e.}, \mathbf{u} \text{ is 1-1 a.e.} \}$

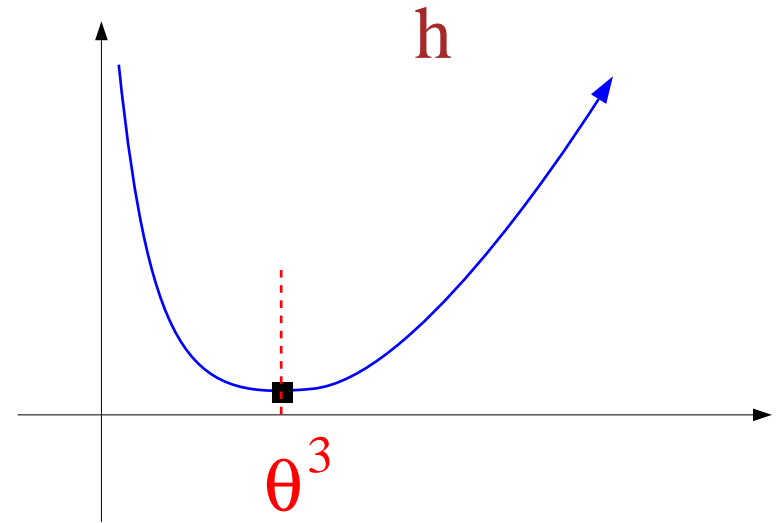
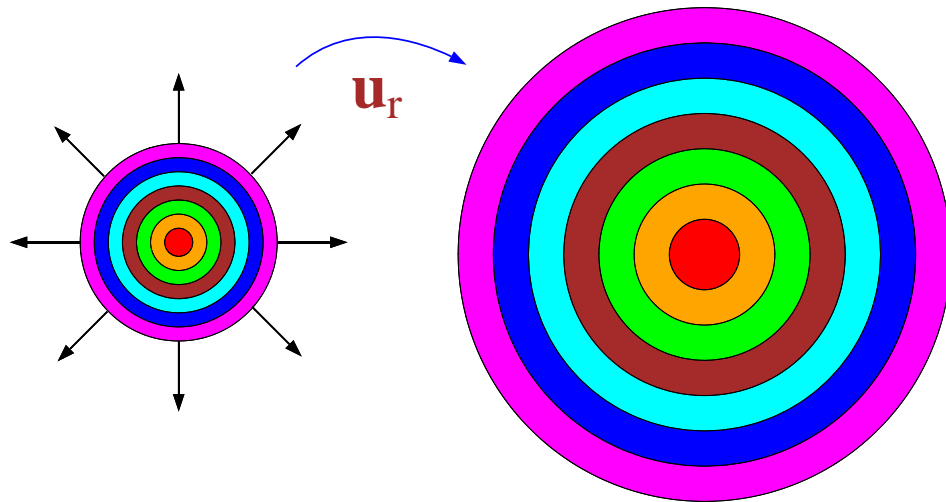
If incompressible then $\det \nabla \mathbf{u} = 1$ a.e.

$\Omega = B := B_1(\mathbf{0}) \dots$ unit ball in \mathfrak{R}^3

$$W(\mathbf{F}) = \alpha |\mathbf{F}|^p + h(\det \mathbf{F})$$

$$\alpha > 0, \quad 1 \leq p < 3$$

h - convex, $h(\tau) \rightarrow +\infty$ as $\tau \rightarrow +\infty$
as $\tau \rightarrow 0^+$



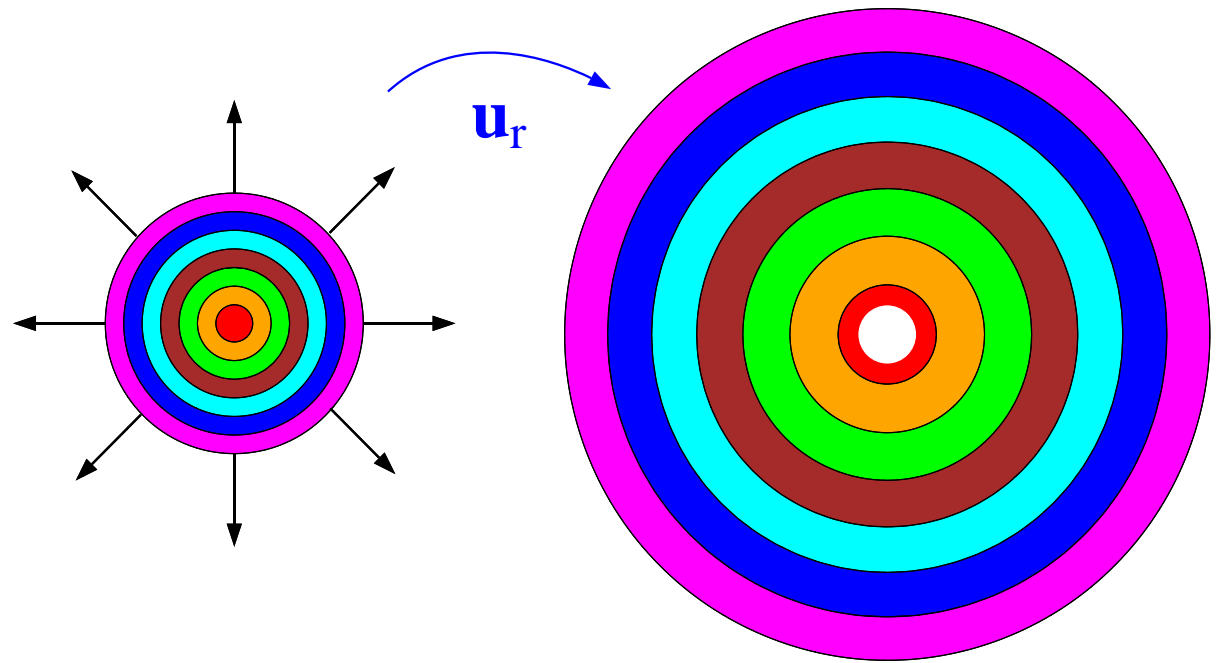
Radial Deformations

$$\mathbf{u}_r(\mathbf{x}) = \frac{r(|\mathbf{x}|)}{|\mathbf{x}|} \mathbf{x}$$

Cavitation: $r(0) > 0$

$$\mathbf{u}_r(\mathbf{x}) = \frac{r(|\mathbf{x}|)}{|\mathbf{x}|} \mathbf{x}$$

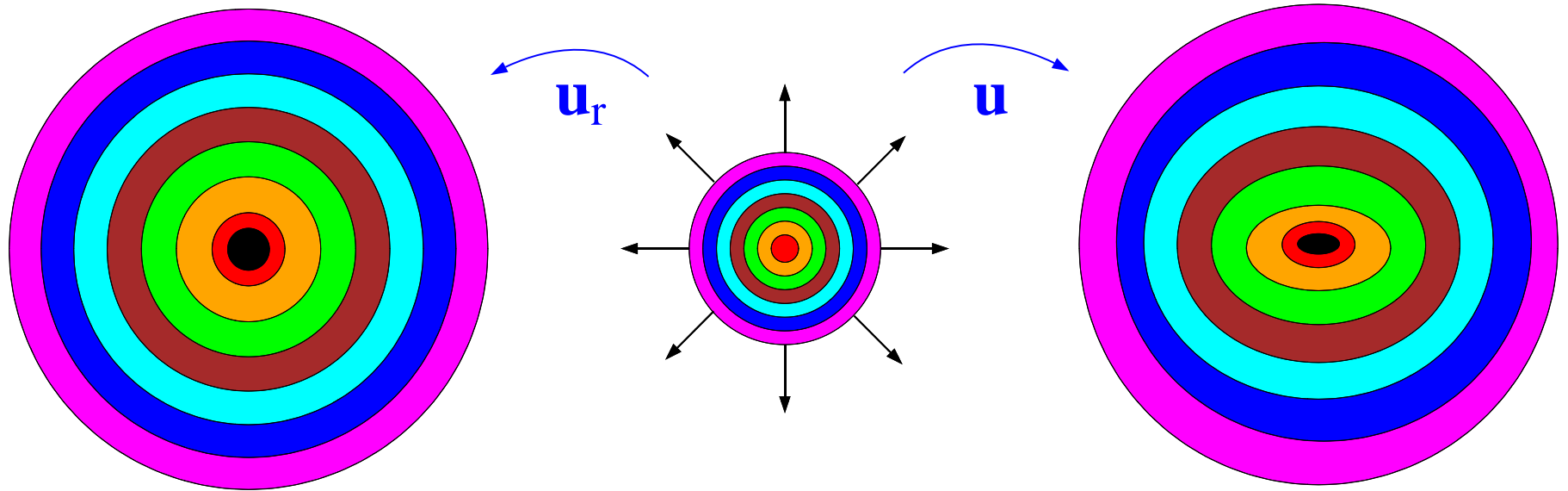
$$r(1) = \lambda$$



Theorem (John Ball). $\exists \lambda_{\text{cr}} \geq \theta$ such that, $\forall \lambda > \lambda_{\text{cr}}$, $\mathbf{v}(\mathbf{x}) = \lambda \mathbf{x}$

does *not* minimize E on $W^{1,p}$. Moreover, the minimizer $\mathbf{u}_r(\cdot, \lambda)$

of E among radial deformations exists and satisfies $r(0) > 0$.



(Simplified) Open Question:

Is the radial minimizer the global minimizer among those deformations that open a single cavity at the center of the ball?

Let \mathbf{u} satisfy $\mathbf{u}(\mathbf{x}) = \lambda \mathbf{x}$ for $\mathbf{x} \in \partial B$.

Then the *radial symmetrization*, \mathbf{u}_r , of \mathbf{u} is given by:

$$\mathbf{u}_r(\mathbf{x}) = \frac{r(|\mathbf{x}|)}{|\mathbf{x}|} \mathbf{x}$$

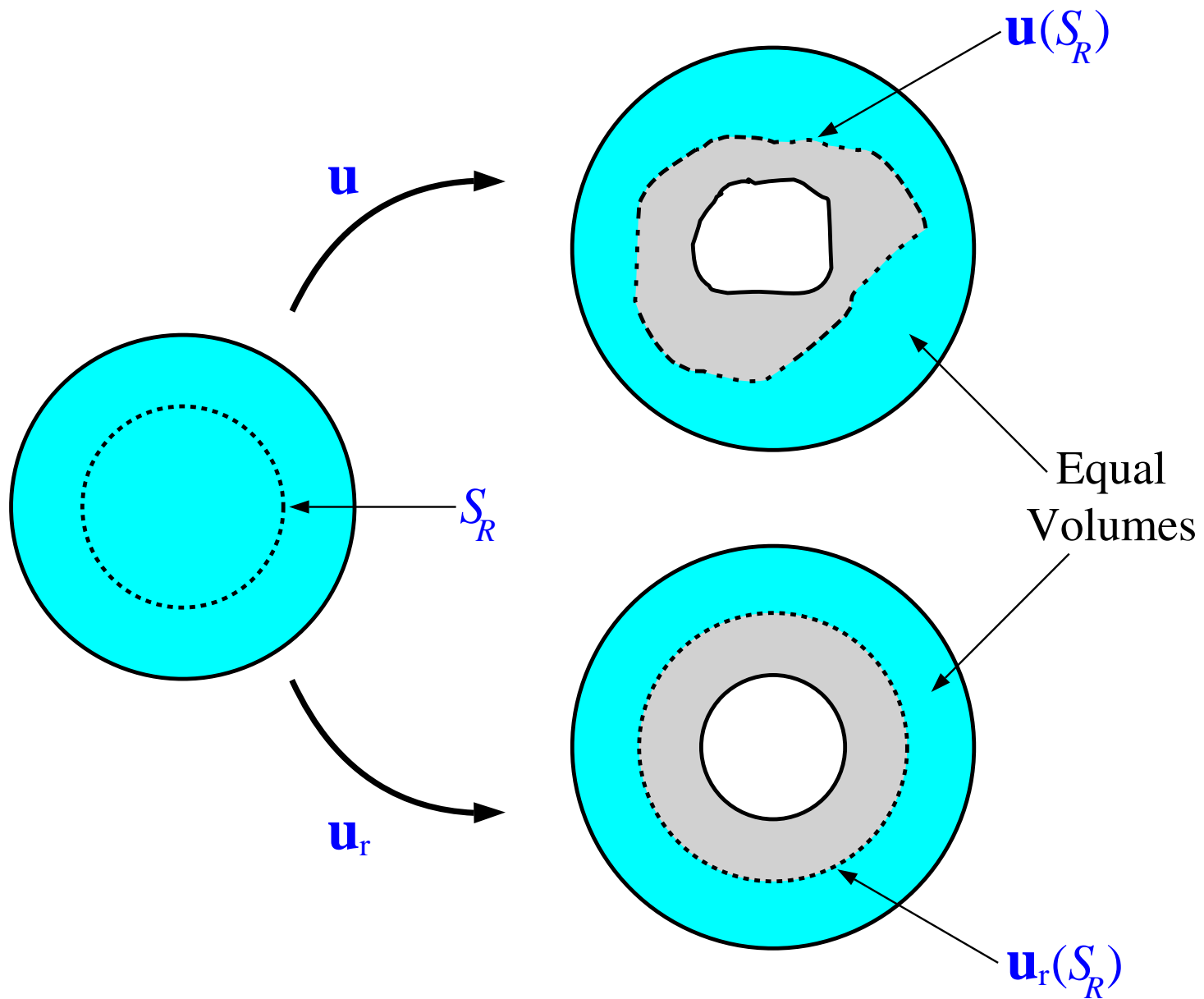
$$\frac{4}{3} \pi r(R)^3 := \frac{4}{3} \pi \lambda^3 - \int_{B \setminus B_R} (\det \nabla \mathbf{u}) d\mathbf{x}$$

Then

$$4\pi r^2 r' = \int_{S_R} (\det \nabla \mathbf{u}) dS_{\mathbf{x}}$$

and hence

$$\int_{S_R} (\det \nabla \mathbf{u}_r) dS_{\mathbf{x}} = \frac{r^2 r'}{R^2} = \int_{S_R} (\det \nabla \mathbf{u}) dS_{\mathbf{x}}$$



Isoperimetric Inequality:

$$\text{Area}(\mathbf{u}(S_R)) \geq \text{Area}(\mathbf{u}_r(S_R))$$

Incompressible Materials

Theorem. Let $\lambda > 1$ and $W(\mathbf{F}) = \Psi(|\mathbf{F}|^p, |\text{adj } \mathbf{F}|^q)$, where

$\Psi \geq 0$ is monotone increasing in each of its arguments and convex.

Suppose that $2 < p < 3$ and $1 < q < 3/2$. Then there exists a

minimizer of E that is radial, among those deformations in the class

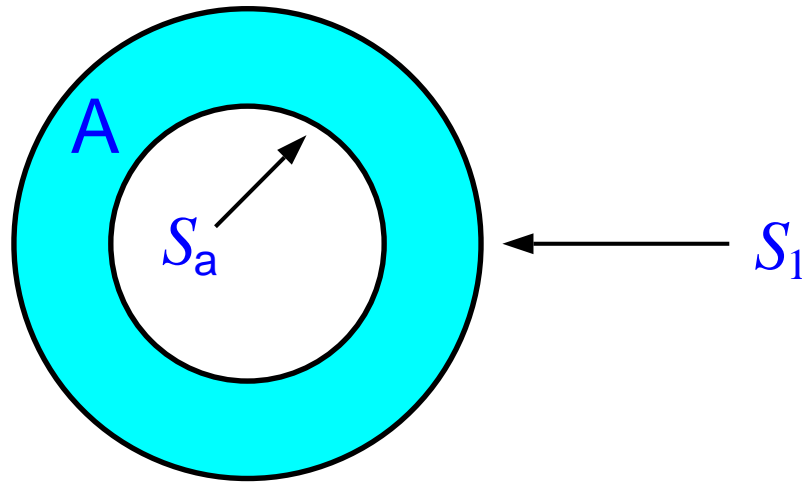
$$\{ \mathbf{u} \in W^{1,1}(B; \mathfrak{R}^3) : \det \nabla \mathbf{u} = 1 \text{ a.e., } \mathbf{u} \text{ is 1-1 a.e., } \mathbf{u} = \lambda \mathbf{i} \text{ on } \partial B \}$$

that open a single hole at the origin. Moreover, if Ψ is strictly convex

then this minimizer is unique.

$$\mathbf{u}_r(\mathbf{x}) = (|\mathbf{x}|^3 + \lambda^3 - 1)^{1/3} \frac{\mathbf{x}}{|\mathbf{x}|}$$

Compressible Materials



A - Annulus

$$\mathbf{u} = \mu \mathbf{i} \text{ on } S_a, \quad \mathbf{u} = \lambda \mathbf{i} \text{ on } S_1$$

Theorem. Let $W(\mathbf{F}) = \Psi(|\mathbf{F}|^p, |\text{adj } \mathbf{F}|^q, \det \mathbf{F})$, where $\Psi \geq 0$ is monotone increasing in each of its first 2 arguments and convex. Suppose that $3 \leq p$ and $3/2 \leq q$. Let $\mu > \lambda$ and $\mu a < \lambda$. Then there exists a minimizer of E that is radial, among those deformations in the class $\{\mathbf{u} \in W^{1,1}(A; \mathfrak{R}^3) : \det \nabla \mathbf{u} > 0 \text{ a.e.}, \mathbf{u} \text{ is 1-1 a.e.}, \mathbf{u} = \mu \mathbf{i} \text{ on } S_a, \mathbf{u} = \lambda \mathbf{i} \text{ on } S_1\}$

An idea from the Proof. Let \mathbf{n} be a unit vector. Then

$$|\det \nabla \mathbf{u}| = |\det \nabla \mathbf{u}| |\mathbf{n}|^2 = (\text{adj } \nabla \mathbf{u})^T \mathbf{n} \cdot (\nabla \mathbf{u}) \mathbf{n} \leq |(\text{adj } \nabla \mathbf{u})^T \mathbf{n}| |(\nabla \mathbf{u}) \mathbf{n}|$$

and hence
$$|(\nabla \mathbf{u}) \mathbf{n}|^2 \geq |\det \nabla \mathbf{u}|^2 / |(\text{adj } \nabla \mathbf{u})^T \mathbf{n}|^2$$

Let $\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}$ be an orthonormal basis. Then

$$\begin{aligned} |\nabla \mathbf{u}|^2 &= |(\nabla \mathbf{u}) \mathbf{t}_1|^2 + |(\nabla \mathbf{u}) \mathbf{t}_2|^2 + |(\nabla \mathbf{u}) \mathbf{n}|^2 \\ &\geq 2 |(\nabla \mathbf{u}) \mathbf{t}_1| |(\nabla \mathbf{u}) \mathbf{t}_2| + |(\nabla \mathbf{u}) \mathbf{n}|^2 \\ &\geq 2 |(\nabla \mathbf{u}) \mathbf{t}_1 \times (\nabla \mathbf{u}) \mathbf{t}_2| + |(\nabla \mathbf{u}) \mathbf{n}|^2 \\ &= 2 |(\text{adj } \nabla \mathbf{u})^T \mathbf{n}| + |(\nabla \mathbf{u}) \mathbf{n}|^2 \geq |(\text{adj } \nabla \mathbf{u})^T \mathbf{n}| + \frac{|\det \nabla \mathbf{u}|^2}{|(\text{adj } \nabla \mathbf{u})^T \mathbf{n}|^2} \end{aligned}$$

$$\int_B |\nabla \mathbf{u}|^2 dx =$$

$$\int_0^1 \int_{S_R} |\nabla \mathbf{u}|^2 dS_{\mathbf{x}} dR \geq \int_0^1 \int_{S_R} \left(|(\text{adj } \nabla \mathbf{u})^T \mathbf{n}| + \frac{|\det \nabla \mathbf{u}|^2}{|(\text{adj } \nabla \mathbf{u})^T \mathbf{n}|^2} \right) dS_{\mathbf{x}} dR$$

Incompressible: $\det \nabla \mathbf{u} \equiv 1$

$$\int_0^1 \int_{S_R} |\nabla \mathbf{u}|^2 dS_{\mathbf{x}} dR \geq \int_0^1 \int_{S_R} \left(|(\text{adj } \nabla \mathbf{u})^T \mathbf{n}| + \frac{1}{|(\text{adj } \nabla \mathbf{u})^T \mathbf{n}|^2} \right) dS_{\mathbf{x}} dR$$

$h(t) = t + 1/t^2$ **is convex** ($p = 2$)

$g(s, t) = t + s^2/t^2$ **is not convex** ($p = 2$)

$k(s, \tau) = (\tau^{2/3} + s^2/\tau^{4/3})^{3/2}$ **is convex** ($p = 3$)

$$(\mathbf{Det} \nabla \mathbf{u})(\varphi) := -\frac{1}{3} \int_{\mathbf{B}} \nabla \varphi \cdot (\text{adj } \nabla \mathbf{u}) \mathbf{u} \, d\mathbf{x}$$

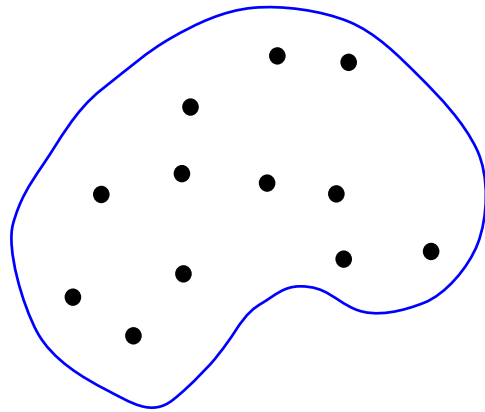
$$\begin{aligned} \text{Div}[(\text{adj } \nabla \mathbf{u}) \mathbf{u}] &= (\text{adj } \nabla \mathbf{u})^T : \nabla \mathbf{u} + \mathbf{u} \cdot \text{Div}[\text{adj } \nabla \mathbf{u}]^T \\ &= 3(\det \nabla \mathbf{u}) + 0 \end{aligned}$$

$$\mathbf{u} \in C^2(\mathbf{B}) \Rightarrow (\mathbf{Det} \nabla \mathbf{u})(\varphi) = \int_{\mathbf{B}} \varphi (\det \nabla \mathbf{u}) \, d\mathbf{x}$$

$$\mathbf{Det} \nabla \mathbf{u} = (\det \nabla \mathbf{u}) \, d\mathbf{x}$$

$$\mathbf{u}^r(\mathbf{x}) = \frac{r(|\mathbf{x}|)}{|\mathbf{x}|} \mathbf{x}, \quad \mathbf{Det} \nabla \mathbf{u}^r = (\det \nabla \mathbf{u}^r) \, d\mathbf{x} + \frac{4}{3} \pi r(0)^3 \delta_0$$

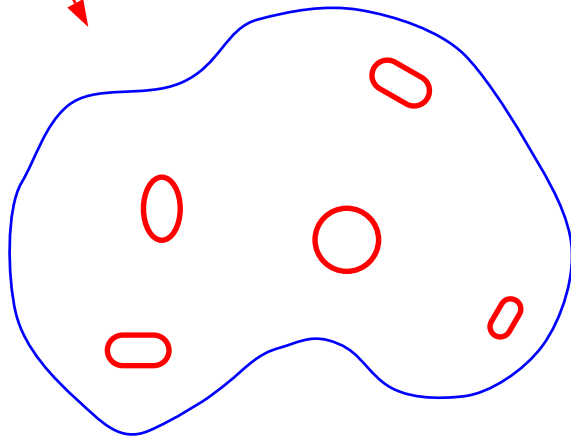
Two Scales



Macroscale: The apparently void-free material will undergo a classical, smooth deformation in response to compression and sufficiently small tensions.



Microscale: Material will undergo a singular, Sobolev deformation in response to a sufficiently large tension.



Today (only) 1. No additional energy associated with cavitation (e.g., no surface energy).

2. Single hole at the center of a ball.