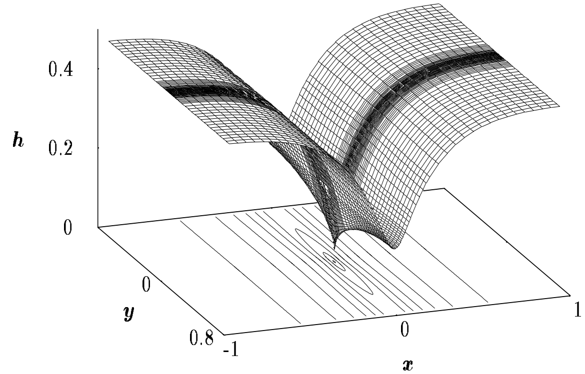
Questions on similarity solutions for rupture: a review



Talk by Tom Witelski (Dept of Math, Duke University)

Research with A. J. Bernoff, A. L. Bertozzi, M. Bowen, M. P. Brenner, K. B. Glasner, M. Gratton, G. Grün, J. R. Lister, A. Münch, D. Vaynblat, B. Wagner

- Solid/Fluid mechanics models with finite-time rupture
- ODEs for the self-similar solutions
- Numerical calculation of solutions
- Form and stability of solutions
- Post-rupture behavior

TPW, *Computing finite-time singularities in interfacial flows.* In A. Bourlioux and M. J. Gander, editors, "**Modern Methods in Scientific Computing and Applications**", NATO ASI proceedings, pp. 451–487. Kluwer, New York, 2002.

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Rupture/Pinch-off dynamics: a concise glossary

Free surface: soln is the evolving boundary of the material domain, $h = h(x,t)$

Topological transition: forming a hole/changing connectivity, $h \to 0$

Finite-time: occurring at a critical time, t_c, behavior for $\tau \equiv t_c - t \to 0$

Singularity: some property of the solution diverges in the limit $\tau \to 0$

Localized: $O(1)$ away from critical position x_c, no divergence is felt as $\tau \to 0$ (see also "focusing")

Self-similar: special classes of solutions (see also "scale invariant")

$$h(x,t) = \tau^{\alpha} \bar{H}(\eta) \qquad \eta = (x - x_c)/\tau^{\beta}$$

Rupture: $\alpha > 0$ Localized: $\beta > 0$

(Blow-up: $\alpha < 0$ Spreading: $\beta < 0$)

Evolution equations: PDE or system of conservation equations determining the motion of the free surface

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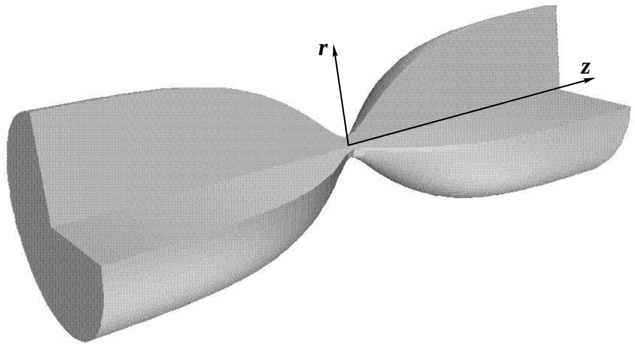
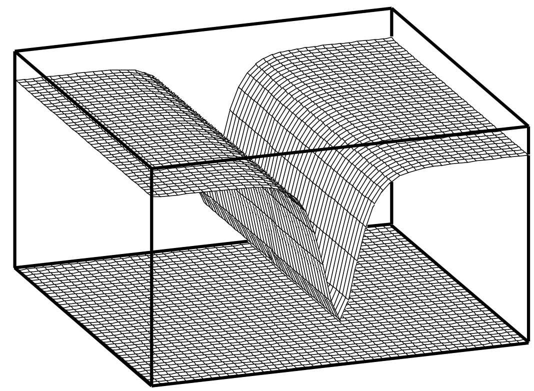
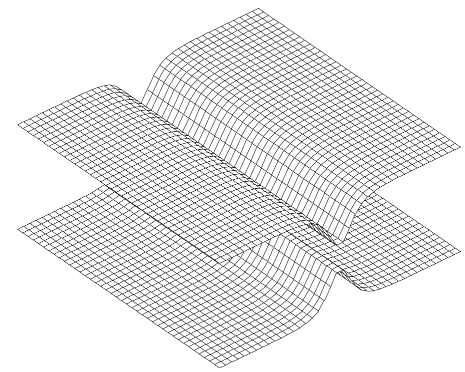
Three representative examples

1. Rupture in surface diffusion of axisymmetric solids [Bernoff, Bertozzi, Witelski]
2. Van der Waals-driven rupture of fluid thin films on solids [Bernoff, Witelski]
3. Generalized vdW-driven rupture of fluid sheets in free space [Vaynblat, Lister, Witelski]

Similarity solutions reduce full models to ODE problems

$$\eta = (x - x_c)/\tau^{\beta} \qquad h(x,t) = \tau^{\alpha}\bar{H}(\eta) \qquad [\quad u(x,t) = \tau^{-\gamma}\bar{U}(\eta) \quad]$$



1. Axisymmetric surface diffusion, $\alpha = \beta = \frac{1}{4}$

$$\frac{1}{4}(\eta\bar{H}' - \bar{H}) = \frac{1}{\bar{H}}\frac{d}{d\eta}\left(\frac{\bar{H}}{\sqrt{1+\bar{H}'^2}}\frac{d}{d\eta}\left[\frac{1}{\bar{H}\sqrt{1+\bar{H}'^2}} - \frac{\bar{H}''}{(1+\bar{H}'^2)^{3/2}}\right]\right)$$

2. Thin film rupture, $\alpha = \frac{1}{5}$, $\beta = \frac{2}{5}$

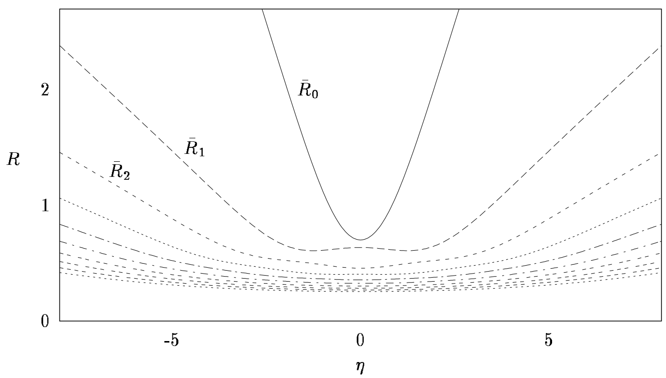
$$\frac{1}{5}(2\eta\bar{H}' - \bar{H}) = -(\bar{H}^{-1}\bar{H}' + \bar{H}^3\bar{H}''')'$$

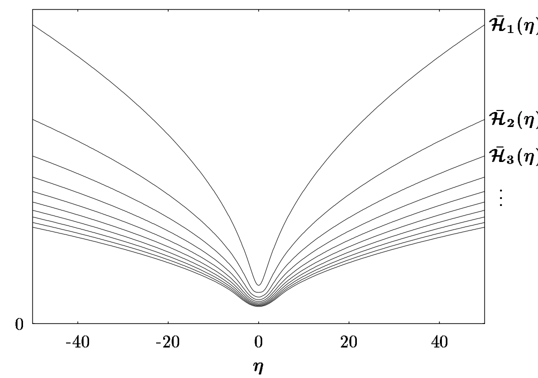
3. Viscous sheet rupture, $\alpha = 1$, $\beta = \gamma = \frac{1}{2}$ (vdW forcing parameter $p > 0$)

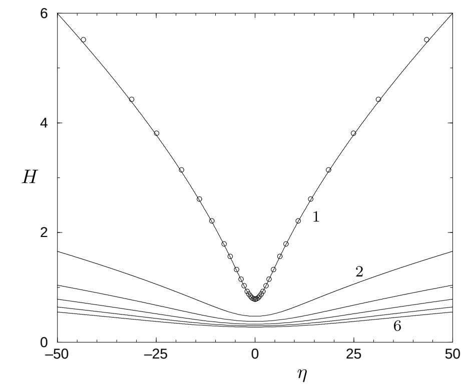
$$\frac{1}{2}\eta\bar{H}' - \bar{H} = -\bar{U}\bar{H}' - \frac{\bar{H}\bar{U}'}{p} \qquad \frac{1}{2}(\eta\bar{U}' + \bar{U}) = -\bar{U}\bar{U}' + \bar{U}'' + \frac{p\bar{H}'\bar{U}'}{\bar{H}} + \frac{\bar{H}'}{\bar{H}^2}$$

Common properties

- First-type solns: scaling exponents $\boldsymbol{\alpha, \beta, \gamma}$ selected by dimensional analysis
- Far-field boundary conditions: $\bar{\boldsymbol{H}} \sim \boldsymbol{C}|\boldsymbol{\eta}|^{\boldsymbol{\alpha/\beta}}$ for $|\boldsymbol{\eta}| \to \infty$ set by *localization* (i.e. matching to the slowly evolving outer region)
- All three have infinite countable sets of plausible solutions $\{\bar{\boldsymbol{H}}_{\boldsymbol{k}}(\boldsymbol{\eta})\}$ $\boldsymbol{k = 1, 2, 3\ldots}$







Discrete sets of localized solutions ("multi-bump solitons") also seen in many other classes of singular rupture/blow-up problems

Q1 Infinite sets of similarity solutions: is this universal or is it a mathematical artefact introduced at some stage of the modeling[a]?

Q1a : Analysis of higher order nonlinear, non-autonomous ODEs and analytical techniques for predicting the structure of the solutions?

[a] E.g. Stokes flow $\longrightarrow$ lubrication equation

Numerical calculations of solutions

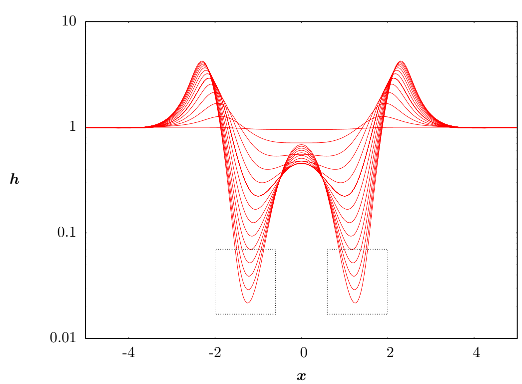
(a) **Shooting methods**: Solve as an initial-value problem to determine ICs that match far-field BCs and give global solutions.
Advantages: Widely used, quick and easy to implement. Yields values for local solution properties ($\bar{H}_{\mathbf{min}}$ is coefficient of rate of thinning)
Weaknesses: Limited by floating point accuracy, easy to miss solutions in higher-order systems.

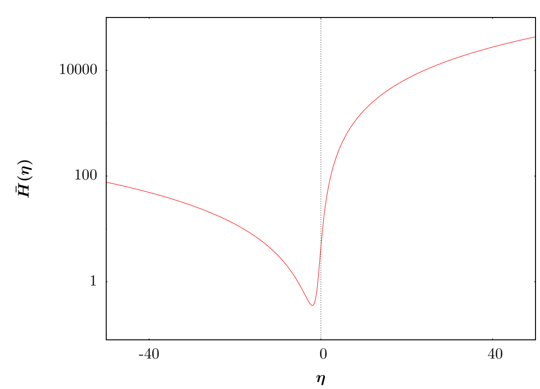
(b) **Global iteration methods**: Discretize the full boundary value problem (finite differences, spectral, FEM). Use Newton-Raphson to converge to soln.
Weaknesses: More work to implement. Good initial guesses needed.
Advantages: Somewhat better global accuracy. Guesses with $H \approx C|\eta|^{\alpha/\beta}$ can be used as a one-parameter ordered set (1-D parameter search space), yields accurate *"nonlinear eigenvalues"* C_k.

- **Issues**: Problems 1 and 2 have been solved using both schemes **[WMVD98 vs. BBW] and [ZL99 vs. BW]** with good agreement. Problem 3 and similar systems have only been solved via shooting **[BLS96, VLW]**.
  Q2 : Can all similarity ODE problems be solved via fixed point iteration?

Form of solutions

- Problems 1 and 2 come from *even*-order nonlinear parabolic PDEs.
  Problem 3 comes from an *odd*-order shallow-water-like dispersive system.
- Problems 1 and 2 are observed to have only symmetric solutions,
  $\bar{H}(-\eta) = \bar{H}(\eta)$. **Q3** : Can this be proven for parabolic PDE on $\mathbb{R}$?
- In problem 3, for different values of parameter p, solutions can be symmetric or asymmetric.
- Issues with asymmetric pinch-off also raised in other viscous vs. inviscid models **[works by J. R. Lister and others]**.
- **Q3a** : Is solution symmetry generally predictable?
- Symmetric perturbations sometimes split to form "book-end" pairs of asymmetric rupture solns separating-off symmetric "satellite drops"





(see also "exploding singularities" **[ABB96]**)

Stability of solutions

Let $s = -\ln(\tau)$ (new time-like variable), then consider

$$h(x,t) = \tau^{\alpha} H(\eta, s) \qquad H(\eta, s) \approx \bar{H}(\eta) + \epsilon \tilde{H}(\eta, s)$$

Linear stability of $\bar{H}(\eta)$ for $s \to \infty$:

$$\tilde{H} = \hat{H}(\eta) e^{\lambda s} \qquad \to \qquad \mathscr{L}\hat{H} = \lambda \hat{H}$$

- Symmetric eigenmodes exist corresponding to symmetries of original PDE, $x_c \to x_c + \epsilon_1$, $t_c \to t_c + \epsilon_2$ (not genuine instabilities)
- In problems 1 and 2 only the first similarity solution, $\bar{H}_1(\eta)$, is stable (numerical calculations).

  Q4 : Are there any physically relevant problems with *multiple* stable solns?

  Q4a : Does existence of unstable solns help in analysis of dynamics from large initial data? Estimates of x_c, t_c?

  Q4b : Analytical methods for stability?

- **Q4c** : Does soln symmetry have implications for stability?

  Q4d : Are there problems with *no* stable similarity solns? The dynamics?

  Q4e : Stability analysis of satellite drops and exploding singularities?

Post-rupture behaviors: continuing-on after singularities?

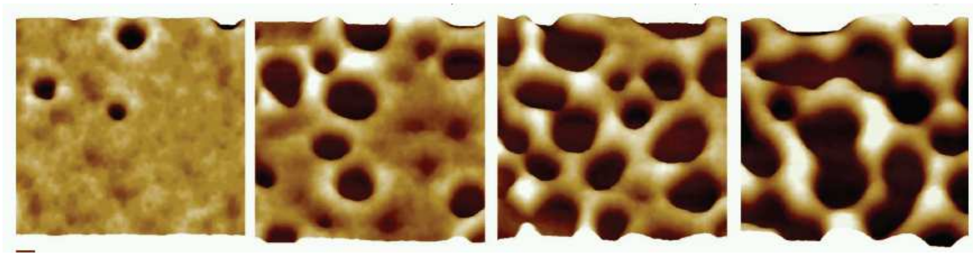
- True-Rupture: change of topology, disconnected independent domains
- Near-Rupture: As $\tau \to 0$, $h \to 0$ but bounded away, $h \geq \epsilon$ by additional physics

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^3 \frac{\partial}{\partial x}\left[\underline{\Pi(h)} - \frac{\partial^2 h}{\partial x^2}\right]\right)$$

Disjoining pressure $\Pi(h)$: Dewetting films on hydrophobic materials

Rupture points $\longrightarrow$ growing "dry spots"

Increasing time $\longrightarrow$ **[Becker et al 2002]**



Dynamics and stability of retreating fluid rims **[works by Münch, Wagner and co.]**

Multi-Rupture pattern formation and later-stage coarsening dynamics **[GW]**

- **Q5** : Are long-time post-rupture dynamics sensitive to behavior at the singularity?

  Q5a : Can simulations of singularity dynamics be eliminated?

  To be replaced by Pre$\longrightarrow$Post singularity transition rules.