Workshop Summary

—T. A. Witten, University of Chicago, July 27, 2008

Introduction

The workshop on Geometric Singularities and Singular Geometries assembled over 85 participants from pure mathmatics, chemical engineering and several other disciplines to discuss a diverse body of phenomena with a common feature: geometric singularities. These are phenomena that take place in physical space and time, an embedding manifold with strong metric properties. The phenomena themselves are physical objects embedded in this manifold. The object might be a fluid, and elastic solid, a liquid crystal or a propagating wave filling space and time out to some distant boundary. It might instead be the interface of such objects, such as the surface of liquid or solid. Or it might be a low-dimensional material body such as a thin fluid film or thread or a solid membrane or rod. All of these objects' basic nature arises from local laws of behavior encoded in differential equations of motion, or a local energy functional. Locality means that the laws of behavior governing any point of the field refer only to neighboring points in the embedding space. Thus the notion of locality depends on the metric structure of space and time. These laws of behavior dictate smoothness of the embedded object or field in the space: smoothly varying streamlines in a fluid, displacement fields in a solid, or the order parameter in a liquid crystal.

Despite this demand for smoothness, an object can often be constrained in such a way that smoothness is lost in a small part of space. The exceptional region is termed a singularity. The demand for smoothness elsewhere and the laws of this smooth behavior typically dictate many features of the singularity, such as its dimensionality and its disposition in space, i. e. its "singular geometry". Often the governing laws can themselves be posed in geometric terms, as in a wave whose wavefronts must be equally spaced or a knot whose defining curves must remain at a fixed distance apart. These are geometric singularities.

The broad occurence of such singularities in many materials, together with the powerful constraints on their behavior dictated by smoothness requirements, are strong motivators for study. Important properties like turbulent drag or plastic flow in solids are attributed to geometric singularities, namely vortices and dislocation lines. A more inclusive understanding of how such singularities can or must behave thus gives power to understand how real materials can and must behave.

Search for common features

The workshop aimed to find common features in the singularities arising in the diverse physical realizations mentioned above. To this end, a series of talks and poster presentations paraded a rich variety of singularities before us. Early in the meeting three schemes were proposed to shape our search for common features. In an introductory talk, the lead organizer Mahadevan pinpointed several specific phenomena of interest. One such phenomenon that appeared prominently in the meeting was *singular minizers* such as creases or voids in a solid under stress and its generalization to fluids and other types of object mentioned above. Mahadevan emphaized descriptions where the appearance of the singular structure could be predicted without reference to those material features that governed its ultimate nonzero size.

A second organizing principle was proposed by Eggers: his focus was on classifying the structure of the object near the singularity. As one approaches a singularity, one expects an asymptotic structure governed by dilation invariance of some kind. Eggers pointed out that a similar dilation invariance occurs near a fixed point of a dynamical system, and that one could anticipate extensive analogies in behavior between the geometric singularities on the one hand and the dynamical singularities on the other. Extending this picture, Kadanoff and Zhang proposed to view the singularities through the lens of information flow. What is the path by which a distant cause leads to an effect at the singularity or vice versa? Information flows not only in the embedding space. It also flows within the state variables that define the material at a given point. Even when this state is described by a single real number in a finite interval, information can flow from tiny segments of this interval to encompass the whole interval. This is the type of information flow in *chaotic* dynamics. In any case, the path of information flow necessarily passes through the dilation invariant vicinity of the singularity; thus information flow is an aspect of the Eggers classification.

Aspects

The existence of singularities like those mentioned above gives rise to a number of natural questions. The most primitive ones concern the form of the singularity itself. What is its *dimensionality* in space and time? How may one characterize its core—by a continuous variable, a discrete one, or some other *group* element? What is its shape in space when it has a nonzero dimensionality? Can the locus of the singularity in the embedding space develop its own *singular subsets*? When several singularites are present how are they mutually disposed in space, *i.e.,* how do singularities *interact*?

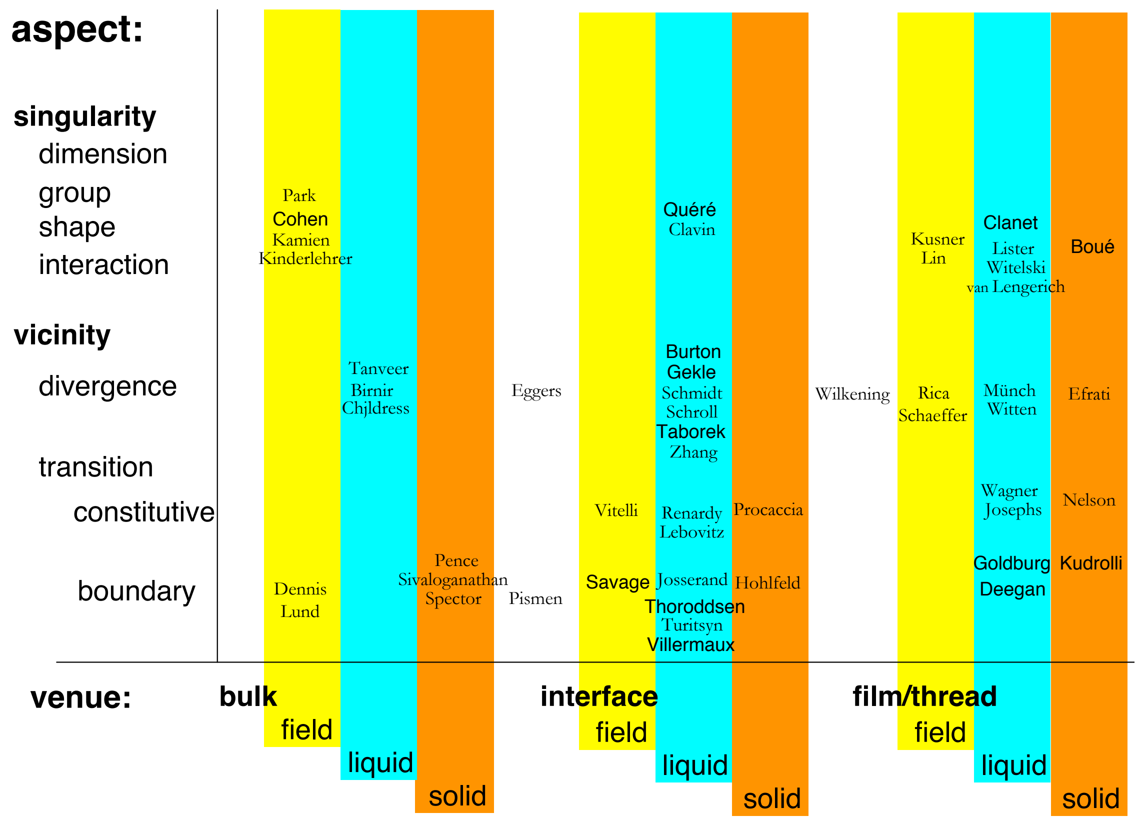
As noted above, the nature of a singularity dictates much about the material or *field* in its vicinity. If x denotes a distance from the singularity and $\psi_x(\Omega)$ denotes some field at a material point at distance x and other co-ordinates Ω, then one expects some form of dilation symmetry in the vicinity of the singularity where $x \to 0$. That is, given a dilation factor λ, one expects $\psi_{\lambda x} = \mathcal{M}_\lambda[\psi_x]$, where $\mathcal{M}$ is some operator on functions of Ω. Further, since the dilation symmetry is valid for an arbitrarily large range of λ, $\mathcal{M}$ can be expressed as a linear operator for differential dilation: $x\partial\psi/\partial x = M\psi$, where M is independent of x. The nature of the linear *dilation operator* M dictates the asymptotic form of ψ_x. If M is a simple multiple of the identity, then ψ has a power-law *divergence* in x. If M is a complex number or equivalent, then the power-law contains oscillation. M can contain a transformation of the Ω manifold such as a rotation, so that lines of corresponding ϕ rotate as they converge to the origin. M may have an important dependence on the function $\psi(\Omega)$, such as a decomposition into eigenspaces. Evidently, when the eigenvalues have different magnitudes, dilation by a large factor amounts to projection of the ψ functions into the subspace of the largest eigenvalue: the *relevant subspace*. Thus understanding of the vicinity or the approach to the singularity amounts to understanding the dilation operator M.

Though M does not depend on the distance from the singularity, it may well depend on other features of the material. It typically depends on the constitutive nature of the material, *i.e.,* the coefficients in the energy or differential equation governing the material. M may also depend on the external conditions such as imposed boundary stress or orientation far from the singularity. Often, continuous changes in these features, do not alter the relevant subspace that dominates near the singularity. Thus the singularity is insensitive to changes of this type; such changes are naturally termed *irrelevant perturbations*. Conversely, some perturbations are relevant and lead to qualitative changes near the singularity. In particular, a given parameter can be irrelevant as it increases up to a certain point, and then cause a *transition* or bifurcation to some new type of singularity.

What emerged?

As hoped, the meeting revealed many of these aspects in a diverse range of phenomena. The specific contributions can be viewed as a network linking the different phenomena presented. The figure attempts to show this network graphically. On the vertical axis are listed the different aspects of geometrical singularities mentioned above. On the horizontal axis are listed the various physical realizations that were treated. Each contribution thus can be represented as a point on the graph. Though this graph gives some sense of the concrete interconnections shown at the meeting, it is necessarily subjective and flawed by incomplete knowledge of the author. It is also nonunique, as there are several ways in which most contributions could be classified. Nevertheless, it is clear from the dense coverage of participants over the figure that the workshop fulfilled well its intention to explore analogous singularities over a broad range of physical systems.

The interactions suggested by the figure were quite evident during the meeting. There was much lively and generous discussion across communities. New ideas for research were born and new contacts for potential collaboration made. On many levels the workshop was a resounding success.



Contributions to the workshop, organized by the *aspect* of the singularity being emphasized and the physical system or *venue* where the singularity occurs. The aspects, listed vertically on the left, are taken from the italicized words in the main text. Names represent participants. Primarily experimental contributions are indicated by sans-serif font. Horizontal position indicates the type of material studied. Vertical position indicates the aspect emphasized. Thus for example, theorist Hohlfield's report on singular kink minimizers at a solid interface induced by imposed stress lies in the orange column for solid interfaces and in the row for boundary-induced transitions,