LECTURE 14

ZIG-ZAG PERSISTENCE

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BOOTSTRAP (B. EFRON '79's)
STUDYING SAMPLE MEANS OR
VARIANCES MORE INFORMATIVE
THAN ANY SINGLE EVALUATION
USEFUL FOR NUMERICAL QUANTITIES
 HOW TO USE THIS IDEA FOR STRUCTURAL INFO RATHER THAN QUANTITATIVE?

SAMPLE, CONSTRUCT

VR(\(-\infty, \infty\)) (FIX \(\varepsilon\))

COMPUTE \(\beta\)

\(S = \text{sample}\)

WRITE \(\beta_1(\text{VR}(S, \varepsilon))\)

\(\beta_0\)
\( \beta_i = 1 \) frequently
Construct Samples \( S_0, S_1, \ldots, S_N \)

\[ S_0 \rightarrow S_0 \cup S_1 \rightarrow S_1 \cup S_2 \rightarrow S_2 \cup S_3 \rightarrow \cdots \rightarrow S_{N-1} \cup S_N \]

"Shape"

Persistence

Apply \( \text{VR}(-; \varepsilon) \), Apply \( N_x \)
Diagram of vector spaces of "zig-zag" shape.
Means that in first case, I expect to see (consistent)

\[ R \rightarrow R \leftarrow R \rightarrow R \rightarrow R \rightarrow R \]

In second case, no consistency between elements

\[ R \rightarrow R \leftrightarrow R \rightarrow R \rightarrow R \rightarrow R \rightarrow R \rightarrow R \]

Is there a way of assessing "consistency"
MUMFORD DATA: $M[t, r] - \text{vary} r$

$M[t, k_0] \neq M[t, k_1], \quad k_0 < k_1 < R$

$M[t, k_0] \circ M[t, k_1] \quad M[t, k_0] \circ M[t, k_2]$

$M[t, k_0] \quad M[t, k_1] \quad M[t, k_2]$

Apply VR(-, ε), and H$

i$
Hope: on can obtain an analysis of behavior over all & by studying (classifying) the possible diagrams of this shape.
WITNESS COMPLEXES

$W(x, \rho, \varepsilon)$ vary $\rho$, see if can confirm that different $\rho$'s represent same phenomenon.

$W(x, \rho, \varepsilon) \not\rightarrow W(x, \rho', \varepsilon)$

NO MAP
RECALL "BIVARIATE" WITNESS $W(x, \alpha, \beta, \varepsilon)$

$W(x, \alpha, \beta, \varepsilon)$

$W(x, \alpha_0, \beta, \varepsilon) \leftarrow W(x, \alpha_0, \alpha, \varepsilon) \rightarrow W(x, \alpha, \beta, \varepsilon) \leftarrow W(x, \alpha_1, \beta, \varepsilon) \rightarrow W(x, \alpha_0, \beta, \varepsilon)$

ZIG-ZAG SHAPE, APPLY $H(\varepsilon^{-1})$ FOR
TIME VARYING DATA $\mathbf{X}$

TIME ASSOCIATED TO EACH PT

$\mathbf{X}[t_0, t_f]$ — subset of $\mathbf{X}$ for which $t \in [t_0, t_f]$

$\mathbf{X}[-R, 1+R] \xrightarrow{\quad} \mathbf{X}[1-R, 1+R] \xrightarrow{\quad} \mathbf{X}[2-R, 2+R]$

ALLOWS E.G., ANALYSIS OF CLUSTER BEHAVIOR OVER TIME.
Can we classify zig-zag diagrams of vector spaces?
Given a directed graph, a representation of $G$ of $G$ is an assignment of vector spaces to $V(G)$, and linear trans. to edges ($e = (v_1, v_2)$, trans. attached to $e$ goes from $V(v_1)$ to $V(v_2)$).
**Graph**

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**Representations**

**Vector Spaces**
- Dimension classified

**Vector Spaces with Self Map**
- Jordan Form

\[ V_0 \xrightarrow{\varphi} V_1 \]

Classified by:
- \( \text{dim}(V_0), \text{dim}(V_1), \text{rank}(\varphi) \)
Graph

Classification
Barcodes from 0 to n persistence?
Wild, field dependent.
P. Gabriel determines all graphs for which classification is tame

= all directed graphs whose underlying undirected graph is a Dynkin diagram
What is the Zig Zag Classification?
Cyclic Modules

\[ 0 \leftarrow R \xrightarrow{1 \times} R \xrightarrow{0!} R \leftarrow 0 \rightarrow 0 \rightarrow 0 \]

Starting Point \rightarrow End Point, Identities between.

Gabriel: Any zig-zag diagram can be written uniquely as a sum of cyclics.
Bar codes classify Zig-Zag diagrams of V-spaces
Similarly, if landmark sets are consistent, expect to see long bars.

More generally, one expects that bar codes will give useful info about time varying, smoothing parameter data.
Study the topology of sets

\[ \{ m \in M \mid R_1 \leq d(M) \leq R_2 \} \]

ZIG-ZAGS PERMIT ONE TO COMPUTE ORDINARY PERSISTENCE MUCH FASTER, AND LESS DEMANDING OF MEMORY RESOURCES.
FINER AND FINER
COVERINGS

TREE-BASED
PERSISTENCE

WHOLE SPACE
Classification can be thought of as module over non-commutative polynomial algebras.

Does this help?
DOING TOPOLOGY ON NON TOPOLOGICAL OBJECTS.

D. QUILLEN: HOMOTOPICAL ALGEBRA

HOMOTOPY THEORY FOR VARIETIES, E.G. OVER FINITE FIELDS?
ÉTALE HOMOTOPY THEORY

NEED ONLY

(a) $\pi_0$ (CONNECTED COMPONENTS)

(b) SYMMETRIES (Galois actions)

$\Rightarrow$ WEIL CONJECTURES ABOUT PT COUNTING ON VARIETIES/TFP

$\Rightarrow$ VOEVOĐSKY, VOEVOĐSKY-SUSLIN
STATISTICS

(1) TIO - CLUSTERING

(2) DEFORMATION / CHANGES OF SCALE

ALGEBRA PERMITS UNDERSTANDING OF
∞ AMTS OF INFORMATION AT ONCE