Lecture 4: A Sequence of Homologies

Homology has three essential ingredients:
1) **Chains**: objects to be counted
2) **Grading**: a notion of "size" or "dimension" for chains
3) **Boundary**: a means of cancelling chains of incident grading, satisfying $d^2 = 0$

This lecture will review several homology theories, emphasizing the variety of types of objects counted.

**Simplicial Homology**

Let $X$ = simplicial complex

- **Chains**: $C_*(X)$ generated by [oriented] simplices of $X$
- **Grading**: dimension of simplex
- **Boundary**: it is what you think... $\partial(\Delta) = \Delta$
  (oriented it need be: $\partial(\Delta) = \Delta$)

**Fast**: $H_* X$ is independent of the simplicial structure on $X$

**Cellular Homology**

Let $X$ = cell complex built from cells of various dimension, in any manner that admits a reasonable notion of inductive gluing...

![Cubical Complex](image)

![Identification Space](image)

![CW-Complex](image)
**SINGULAR HOMOLOGY**

Let $X$ = any topological space.

**Chains**: $C_\ast^\text{sing} X$ generated by maps of [oriented] simplices into $X$

$$\sigma = \left( \bigtriangleup \xrightarrow{\sigma} X \right)$$

the Platonic 2-simplex

**Grading**: dimension of the domain of $\sigma$, not its image in $X$

**Boundary**: the boundary of a map of a simplex is the induced maps on the faces...

$$\partial \left( \bigtriangleup \xrightarrow{\sigma} X \right) = \left( \Delta \xrightarrow{\partial} X \right)$$

**Problem**: $C_\ast X$ is no longer finitely generated, unless $X$ is a finite set.

**However**: $C_\ast X$ is so large that it contains e.g., all possible simplicial structures on $X$, as well as all possible deformations thereof.

**Two Foundational Results**:

**Theorem**: $H_\ast^\text{sing} X$ is an invariant of homotopy equivalence.

This is a reasonable result given the sheer size of $C_\ast^\text{sing}$. The key tool in this proof is an algebraic device known as a "Second Homology".

**Theorem**: $H_\ast^\text{sing} \cong H_\ast^\text{cell}$ whenever the space has a well-defined cellular (or simplicial) structure.

This uses induction on dimension, along with an algebraic tool called the "5-Lemma".
ČECH HOMOLOGY

Let \( \mathcal{U} = \{ U_\alpha \} \) be a cover of \( X \) by open sets.

**Chains:** non-empty intersections of \( \{ U_\alpha \} \)

**Grading:** "depth" of intersection

This is a generator of \( 
\widetilde{C}_2 \)

**Boundary:**

\[
\partial ( \bigcap U_\alpha ) = \bigcap ( \partial U_\alpha )
\]

\[
\partial ( \bigcap U_\alpha ) = \bigcap ( \partial U_\alpha )
\]

(where the +/− are determined by an ordering on the index set \( \alpha = \{1, 2, \ldots, n\} \))

The signs on the boundary operator \( \partial : \tilde{C}_n \to \tilde{C}_{n-1} \) are arranged so that \( \partial \circ \partial = 0 \) and the resulting homology is well-defined.

**Theorem:** If \( U \) is acyclic -- if \( H_\ast (U_\alpha \cap \cdots \cap U_\omega) \cong H_\ast (\text{point}) \) for all non-empty intersections \( U_\alpha \cap \cdots \cap U_\omega \), then \( H_\ast = H_\ast (\bigcup U_\alpha) \).

One often passes through an intermediate simplicial structure known as the nerve of the cover -- a \((k-1)\) simplex of \( N(\mathcal{U}) \) is a non-empty intersection of \( k \) distinct elements of \( \mathcal{U} \).

The nerve lemma of Leray says that if all nonempty intersections in \( \mathcal{U} \) are contractible sets, then \( N(\mathcal{U}) \) is homotopy equivalent to \( \bigcup U_\alpha \).
MORSE HOMOLOGY

The structure of this homology theory is quite different. Let $X$ be a smooth, compact manifold. Morse theory studies the topology of $X$ as it relates to the calculus of functionals $f: X \to \mathbb{R}$.

- Choose a (Riemannian) metric for $X$
- Choose a smooth function $f: X \to \mathbb{R}$ ("height function")
- Critical points of $f$ are points of $X$ on which $\nabla f = 0$
- $f$ is a Morse function if all critical points are nondegenerate

**Aside**

**Hessian:** $H_f_p = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{i,j}$ is matrix of 2nd derivatives of $f$

$p$ is a non-degenerate critical point if $Df_p = 0$ & $H_f_p$ has no zero eigenvalues.

The unstable manifold of a critical point $p$ is the set of points that converge to $p$ under the gradient flow as time $\to -\infty$

$W^u(p) = \{ x(t) : t \to -\infty, x(0) = p \}$

**Chains:** Generators of $MC_\ast(f)$ are critical points of $f$

**Grading:** Morse index of $p \in \text{Crit}(f)$ is

$\text{M}(p) = \# \text{ negative eigenvalues of } H_f_p$

$= \dim W^u(p)$

**Intuition:** The Morse index is the "degree of instability" of the critical point under the flow of $-\nabla f$.

**Boundary:** In $\mathbb{Z}_2$ coefficients, $\text{M}$ counts mod 2 the number of flowlines of $-\nabla f$ which connect critical points of incident $\text{M}$. 
It's best work mod 2 and do an example:

\[ \mathbb{M}_x^f(f) \] has 6 generators:

\[ \cdots \rightarrow 0 \rightarrow F_A \oplus F_B \rightarrow F_C \oplus F_D \rightarrow F_E \oplus F_F \rightarrow 0 \]
\[ \mu = 2 \quad \mu = 1 \quad \mu = 0 \]

\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

Here, \( F = \mathbb{Z}_2 \) = field of 2 elements...

The homology of \( f \) is illustrative:

\[ MH_2(f) = \ker \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \cong F \text{ with generator } \begin{bmatrix} A + B \end{bmatrix} \]

\[ MH_1(f) = \ker \begin{bmatrix}
0 & 1
\end{bmatrix} / \img \begin{bmatrix}
1 & 1
\end{bmatrix} \cong 0 \]

\[ MH_0(f) = F_1 \oplus F_2 / \img \begin{bmatrix}
0 & 1
\end{bmatrix} \cong F \text{ with } \begin{bmatrix} E \end{bmatrix} = \begin{bmatrix} F \end{bmatrix} \text{ as generators} \]

\[ MH_*(f) \cong H_*^{\text{sing}}(X) \] is no coincidence.

**Theorem:** For \( X \) a compact manifold, \( MH_*(f) \cong H_*(X; \mathbb{Z}_2) \)

The proof of this theorem is not so mysterious: the isomorphism comes via cellular homology, using unstable manifolds as a cell structure on \( X \):

\[ p \in \text{CRIT}(f) \mapsto W^u(p) \]

\[ MH_*(f) \overset{\cong}{\rightarrow} H_*(X) \overset{\cong}{\rightarrow} H_*(X) \]

I've hidden a few details: one needs to perturb \( f \) and/or the gradient operator (via the metric) to ensure a good \( D \)-operator. On the other hand, I've not told you about Conley-Morse theory, which counts fairly arbitrary invariant sets of arbitrary continuous vector field, on an arbitrary locally-compact metric space — one can dispense with the usual “nondegenerate” conditions.
**Combinatorial Morse Homology**

The fact that Morse homology is related to singular cellular homology should be a hint that one can count more general notions of "critical objects" with "flowlines" as a boundary operator. Indeed several current research projects (Fiber theories, contact homology, symplectic field theory, ...) rely on this principle. Here is a simple, combinatorial instantiation:

Let $X$ = simplicial or cell complex

A COMBINATORIAL VECTOR FIELD on $X$ is a collection $V$ of pairs

$V = \{ (T_x, \sigma_x) \mid T_x$ is a face of $\sigma_x$, and each cell of $X$ lies in at most one pair. A CRITICAL CELL of $X$ is one not listed in $V$. 

- **Chains**: generated by critical cells of $V$
- **Grading**: dimension of cells

A FLOWLINE of $V$ is a sequence of $V$-paired cells

$T_0 < \sigma_0 \rightarrow T_1 < \sigma_1 \rightarrow T_2 < \sigma_2 \rightarrow \cdots \rightarrow T_m < \sigma_m \rightarrow T_n$

"is a face of" in $V$

$V$ is a GRADIENT FIELD if there are no flowlines with $T_{n+1} = T_0$.

MC*($V$) generated by critical cells of a gradient combinatorial field $V$

$\mathcal{Z} : MC_k(V) \rightarrow MC_{k-1}(V)$ counts the number of paths (mod 2, let's say)

$\sigma > T_0 < \sigma_0 \rightarrow T_1 < \sigma_1 \rightarrow T_2 < \sigma_2 \rightarrow \cdots \rightarrow T_m < \sigma_m \rightarrow T_n$

**Theorem**: $MH_k(V) = H^c_k(X)$, independent of $V$

This is one piece of evidence that Morse theory does not really need all those conditions about smooth manifolds, nondegenerate Hessians, etc. For more on this subject, see papers of Forman & recent text of Koslov.
Given a multiplicity of (largely) equivalent homology theories, one proceeds to play one theory off another for gain:

**Lemma:** If \((C_*, d)\) is a finite-dimensional chain complex and \(H_*\) its homology, then

\[
\chi(C_*) = \sum_{k=0}^{\infty} (-1)^k \dim C_k = \sum_{k=0}^{\infty} (-1)^k \dim H_k = \chi(H_*)
\]

**proof:** Via linear algebra & the definitions:

1) \(\dim C_n = \dim \mathbb{Z}_n + \dim \mathbb{B}_n\)
2) \(\dim \mathbb{Z}_n = \dim \mathbb{B}_n + \dim H_n\)

Thus,

\[
\chi(C_*) = \sum_{k} (-1)^k (\dim \mathbb{Z}_k + \dim \mathbb{B}_{k-1}) \quad \text{via (1)}
\]

\[
= \sum_{k} (-1)^k (\dim H_k + \dim \mathbb{B}_k + \dim \mathbb{B}_{k-1}) \quad \text{via (2)}
\]

\[
= \sum_{k} (-1)^k \dim H_k = \chi(H_*) \quad \text{telescoping sum}
\]

**QED.**

**Cor:** The Euler characteristic is a homology invariant of finite cell complexes.

**proof:** If \(X\) and \(Y\) are homology equivalent finite cell complexes, then

\[
\chi(X) = \chi(C^\text{cell}_* X) = \chi (H^\text{cell}_* X) = \chi (H^\text{sub}_* X) = \chi (H^\text{sing}_* Y) = \chi (H^\text{cell}_* Y) = \chi(C^\text{cell}_* Y) = \chi(Y)
\]

**Theorem:** [Poincaré Duality] For \(X\) a compact manifold of dimension \(n\),

\[
H_k(X; \mathbb{R}) \cong H_{n-k}(X; \mathbb{R})
\]

**proof:**

\[
\begin{align*}
H_k(X; \mathbb{R}) &\cong \text{MH}_k(f) \cong \text{MH}_k(-f) \cong \text{MH}_{n-k}(f) \cong H_{n-k}(X; \mathbb{R}) \\
\end{align*}
\]

**QED.**
Cor: A compact connected manifold $\mathcal{X}$ has a FUNDAMENTAL CLASS, a generator of $H_{\dim(\mathcal{X})}(\mathcal{X}, \mathbb{Z}_2)$.

proof: $\dim H_{\dim(\mathcal{X})}(\mathcal{X}, \mathbb{Z}_2) = \dim H_0(\mathcal{X}, \mathbb{Z}_2) = 1$  

$\uparrow$ duality $\uparrow$ connectivity

This fundamental class, $[\mathcal{X}] \in H_{\dim(\mathcal{X})}(\mathcal{X}, \mathbb{Z}_2)$, is the basis of INTERSECTION THEORY

Cor: Any compact odd-dimensional manifold has $\chi = 0$.

proof: $\chi(\mathcal{X}) = \sum_k (-1)^k \dim H_k(\mathcal{X}, \mathbb{Z}_2)$

$= \sum_k (-1)^k \dim H_{n-k}(\mathcal{X}, \mathbb{Z}_2)$ $\quad n = \dim \mathcal{X}$

$= \sum_k (-1)^{n-k} \dim H_{n-k}(\mathcal{X}, \mathbb{Z}_2)$

$= (-1)^n \chi(\mathcal{X})$  

$\Box$