An Introduction to Multidimensional Conservation Laws Part III: Divergence-Measure Vector Fields and Entropy Solutions without Bounded Variation

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Integration by Parts (Leibniz, Oct. 29, 1675, based on the fundamental theorem of Calculus by Newton 1669; also Barrow 1630-77 and Gregory 1638-75):

Let $f(y), g(y) \in C^1(\mathbb{R})$. Then, for any $a \leq b$,

$$\int_a^b f(y)g'(y)\,dy = \big(f(b)g(b) - f(a)g(a)\big) - \int_a^b f'(y)g(y)\,dy.$$

Gauss-Green Theorem (Divergence Theorem): Let $\Omega \Subset \mathcal{D} \subset \mathbb{R}^N$ be compact and have a piecewise smooth boundary. If $\mathbf{F} \in C^1(\mathcal{D}; \mathbb{R}^N)$, then

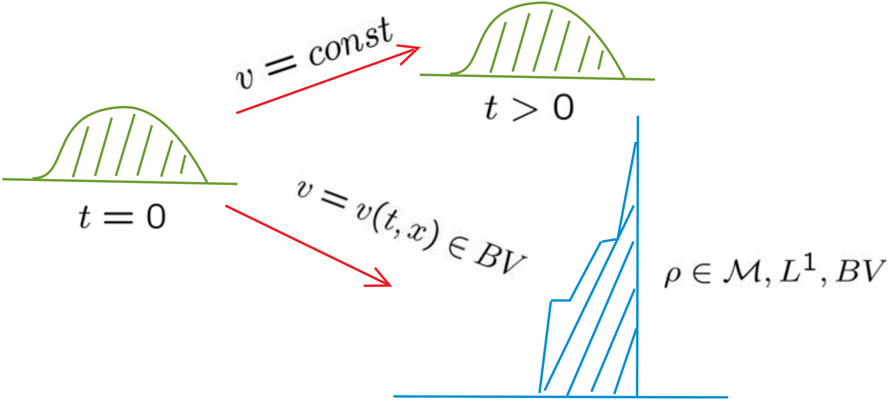
$$\int_\Omega \varphi\, \mathrm{div}\mathbf{F}\, d\mathbf{y} = -\int_{\partial\Omega} \varphi\, \mathbf{F}\cdot\nu\, dS - \int_\Omega \mathbf{F}\cdot\nabla\varphi\, d\mathbf{y}$$

for any $\varphi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$, where ν is the unit interior normal on $\partial\Omega$ to Ω and dS is the surface measure (Carl Friedrich Gauss in 1813, George Green in 1825).

Achievements of 20th Century: **Sobolev Spaces, BV Space, $\cdots$**

Traces, Gauss-Green formula, $\cdots$

Transport Equation: $\partial_t \rho + \partial_x (v\rho) = 0$

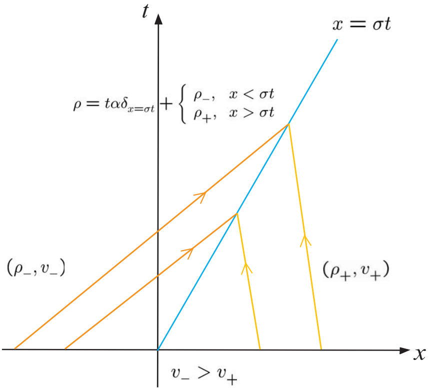


$$\int \rho(t, x)dx = \int \rho(0, x)dx$$

* ρ-density, v-velocity

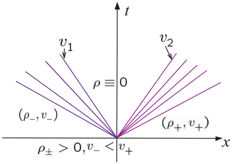
Pressureless Euler Equations

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2) = 0$$



$$\alpha = \frac{1}{\sqrt{1+\sigma^2}}(\sigma[\rho] - [\rho v]) > 0, \qquad \sigma = \frac{\sqrt{\rho_+}v_+ + \sqrt{\rho_-}v_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} \in (v_+, v_-)$$

$$\partial_t \rho + \partial_x(\rho v) = 0, \qquad \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0$$



Pressure Function $p(\rho) = \kappa\rho^\gamma$

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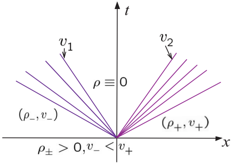
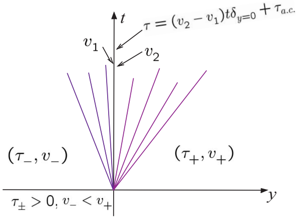


$$(t,x) \to (t,y):\ y_t = \rho(t,x), y_x = -(\rho v)(t,x); \qquad \tau(t,y) = 1/\rho(t,x)$$



$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0$$

$\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$

$\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_d) : \mathbb{R}^m \to (\mathbb{R}^m)^d$ is a nonlinear mapping

$\mathbf{f}_i : \mathbb{R}^m \to \mathbb{R}^m$ for $i = 1, \ldots, d$

$$\partial_t \mathbf{A}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) + \nabla \cdot \mathbf{B}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) = 0$$

Connections and Applications:

- Fluid Mechanics and Related: Euler Equations and Related Equations
  Gas, shallow water, elastic body, combustion, MHD, ...
- Differential Geometry and Related: Gauss-Codazzi Equations and Related Equations Embedding, Emersion, ...
- Relativity and Related: Einstein Equations and Related Equations
  Non-vacuum states, ...
- ......

Cauchy Problem:

$$\begin{cases} \partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \\ \mathbf{u}\big|_{t=0} = \mathbf{u}_0(\mathbf{x}) \end{cases}$$

Challenges: Singularity $\rightarrow$Discontinuous/Singular Solutions

- Shock Waves, Vortex Sheets, Vorticity Waves, ...
- Focusing and Breaking of Waves, ...
- Concentration, Cavitation, ...
- ......

Posed Spaces for Entropy Solutions ??

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Posed Spaces for Entropy Solutions ??

Candidates: BV, L^∞, L^p, $\mathcal{M}$, ...

BV Space: Well-Posedness Space for Entropy Solutions?

1-D: Glimm's Theorem (1965): $\|\mathbf{u}(t,\cdot)\|_{BV} \le C\,\|\mathbf{u}_0\|_{BV}$

L^1-**Stability**: $\|\mathbf{u}(t,\cdot)-\mathbf{v}(t,\cdot)\|_{L^1(\mathbb{R})} \le C\,\|\mathbf{u}(0,\cdot)-\mathbf{v}(0,\cdot)\|_{L^1(\mathbb{R})}$

Bressan et al, Liu-Yang, Bianchini-Bressan, LeFloch, ...

Strictly hyperbolic systems, initial data of small BV: Works!

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(K. Jenssen, $\cdots\cdots$)

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Multi-D: **(?)** $\|\mathbf{u}(t,\cdot)\|_{BV} \le C\,\|\mathbf{u}_0\|_{BV}$ (*)

Rauch (1986): A necessary condition for (*) is

$$\nabla \mathbf{f}_k(\mathbf{u})\nabla \mathbf{f}_l(\mathbf{u}) = \nabla \mathbf{f}_l(\mathbf{u})\nabla \mathbf{f}_k(\mathbf{u}) \qquad \text{for all } k,l=1,2,\dots,d$$

Special cases: $m=1$ or $d=1$

$\mathbf{f}_k(\mathbf{u}) = \phi_k(|\mathbf{u}|^2)\mathbf{u}, \qquad k=1,2,\dots,d$

2003: Ambrosio-De Lellis, Bressan: (*) fails

2005: De Lellis: Blowup of $\|\mathbf{u}(t,\cdot)\|_{BV}$ in finite time

Entropy Solutions in L^∞, L^p_w, and $\mathcal{M}$

- $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M}(\mathbb{R}^{d+1}_+)$ **or** $L^p_w(\mathbb{R}^{d+1}_+), \quad 1 \le p \le \infty$**;**
- **For any convex entropy-entropy flux pair** $(\eta, \mathbf{q})$ **so that** $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))(t,\mathbf{x})$ **is a distributional field,**

$$\mu_\eta := \partial_t \eta(\mathbf{u}) + \nabla \cdot \mathbf{q}(\mathbf{u}) \le 0$$

in the sense of distributions (i.e. $(\eta, \mathbf{q}) := (\eta, q_1, \dots, q_d)$ **is a solution of** $\nabla q_k(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}), 1 \le k \le d$**).**

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Examples: $\mathbf{u}(t,\mathbf{x}) \in L^\infty, L^p$ by Compensated Compactness
Isentropic Euler Equations, ...

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Schwartz Lemma $\Rightarrow$ $\mathrm{div}_{(t,\mathbf{x})}(\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x}))) \in \mathcal{M}(\mathbb{R}^{d+1}_+)$.

Observation: When $\mathbf{u} \in L^\infty$, this is also true for any C^2 entropy pair $(\eta, \mathbf{q})$ (η not necessarily convex) if the system has a strictly convex entropy (cf. Chen 1991).

Divergence-Measure Fields over an open set $\mathcal{D} \subset \mathbb{R}^N$

- For $1 \le p \le \infty$, $\mathbf{F}$ is called a $\mathcal{DM}^p(\mathcal{D})$-field if $\mathbf{F} \in L^p(\mathcal{D})$ and

$$\|\mathbf{F}\|_{\mathcal{DM}^p(\mathcal{D})} := \|\mathbf{F}\|_{L^p(\mathcal{D};\mathbb{R}^N)} + \|\mathrm{div}\,\mathbf{F}\|_{\mathcal{M}(\mathcal{D})} < \infty; \tag{1}$$

- The field $\mathbf{F}$ is called a $\mathcal{DM}^{ext}(\mathcal{D})$-field if $\mathbf{F} \in \mathcal{M}(\mathcal{D})$ and

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$\mathcal{DM}^p(\mathcal{D})$ and $\mathcal{DM}^{ext}(\mathcal{D})$ are **Banach spaces**, which are LARGER than the space of BV fields (they coincide when $N = 1$).

BV theory (esp. the Gauss-Green Formula and Traces) has significantly advanced our understanding of solutions of nonlinear PDEs and related problems in the calculus of variations, differential geometry,...

Goal:

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Goal: Develop a $\mathcal{DM}$ theory to deal with entropy solutions without bounded variation for nonlinear conservation laws and related problems

Examples

1: $\mathbf{F}(y_1, y_2) = (\sin\left(\frac{1}{y_1 - y_2}\right), -\sin\left(\frac{1}{y_1 - y_2}\right))$.

(i) $\mathbf{F} \in \mathcal{DM}^\infty(\mathbb{R}^2)$, while $F_j \notin BV(\mathbb{R}^2)$ for $j = 1, 2$;

(ii) $\mathbf{F}$ has an essential singularity at each point of $L = \{y_1 = y_2\}$, therefore, $\mathbf{F}$ has no trace on L in the classical sense.

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However, for $\Omega = \{\mathbf{y} : |\mathbf{y}| < 1, y_2 > 0\}$,

$$\int_\Omega \operatorname{div} \mathbf{F} = 0 \neq -\int_{\partial\Omega} \mathbf{F} \cdot \nu \, d\mathcal{H}^1 = \pi \quad \text{(in the classical sense)},$$

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3: For any $\mu_i \in \mathcal{M}(\mathbb{R}), i = 1, 2$, with finite total variation,

$$\mathbf{F}(y_1, y_2) = (\mu_1(y_2), \mu_2(y_1)) \in \mathcal{DM}^{ext}(\mathbb{R}^2).$$

A non-trivial example of such fields is provided by the Riemann solutions of the 1-D Euler equations in Lagrangian coordinates for which the vacuum generally develops.

Sets of Finite Perimeter $E \subset \mathbb{R}^N$: $\chi_E \in BV(\mathbb{R}^N)$

For every $\alpha \in [0,1]$, define the set of all points with density α:

$$E^\alpha := \{\mathbf{y} \in \mathbb{R}^N : \lim_{r\to 0} \frac{|E \cap B(\mathbf{y},r)|}{|B(\mathbf{y},r)|} = \alpha\}.$$

E^0—Measure-theoretic Exterior, E^1—Measure-theoretic Interior

$\partial^m E := \mathbb{R}^N \setminus (E^0 \cup E^1)$—Measure-theoretic Boundary

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Reduced Boundary $\partial^* E$: Set of all points $\mathbf{y} \in \Omega$ such that

(i) $\|\nabla \chi_E\|\,(B(\mathbf{y},r)) > 0$ for all $r > 0$;

(ii) The limit $\nu_E(\mathbf{y}) := \lim_{r\to 0} \frac{\nabla\chi_E(B(\mathbf{y},r))}{\|\nabla\chi_E\|(B(\mathbf{y},r))}$ exists.

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$\Rightarrow$ (i) For $\mathcal{H}^{N-1}$-a.e. $\mathbf{y} \in \partial^* E$, $\lim_{r\to 0} \frac{\|\nabla\chi_E\|(B(\mathbf{y},r))}{\alpha(N-1)r^{N-1}} = 1$;

(ii) $\|\nabla\chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^* E$.

$\boldsymbol{\nu}_E(\mathbf{y})$ (unit vector)–measure-theoretic interior unit normal to E at $\mathbf{y}$

- $\partial^* E \subset E^{\frac{1}{2}} \subset \partial^m E$; $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$.

Mollification: $u_\varepsilon := u * \rho_\varepsilon$ for $u \in L^1(\mathbb{R}^N)$, $\rho_\varepsilon(\mathbf{y}) := \frac{1}{\varepsilon^N}\rho(\frac{\mathbf{y}}{\varepsilon})$ with $\rho \in C_c^\infty(\mathbb{R}^N), \rho \geq 0, spt(\rho) \subset \{|\mathbf{y}| \leq 1\}, \|\rho\|_{1;\mathbb{R}^N} = 1$

- For each $\varepsilon > 0$, $u_\varepsilon \in C^\infty(\mathbb{R}^N)$ *and* $D^\alpha(\rho_\varepsilon * u) = (D^\alpha \rho_\varepsilon) * u$;
- $u_\varepsilon(\mathbf{x}) \to u(\mathbf{x})$ *whenever* $\mathbf{x}$ *is a Lebesgue point of* u. In particular, if $u \in C(\mathbb{R}^N)$, u_ε *converges uniformly to* u *on compact subsets of* $\mathbb{R}^N$.

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When $u = \chi_E$ for a set of finite perimeter, E, and u_ε is the mollification of χ_E, we have the following stronger results, besides the above properties:

- There is a set $\mathcal{N}$ with $\mathcal{H}^{N-1}(\mathcal{N}) = 0$ and a function $u_{_E} \in BV$ such that, for all $y \notin \mathcal{N}$, $u_\varepsilon(y) \to u_{_E}(y)$ as $\varepsilon \to 0$ and

$$u_{_E}(y) = \begin{cases} 1 & y \in E^1, \\ \frac{1}{2} & y \in \partial^* E, \\ 0 & y \in E^0; \end{cases}$$

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- $\nabla u_\varepsilon \overset{*}{\rightharpoonup} \nabla u_{_E}$ in $\mathcal{M}(\mathbb{R}^N)$, and $\nabla \chi_E = \nabla u_{_E}$;

Mollification: $u_\varepsilon := u * \rho_\varepsilon$ for $u \in L^1(\mathbb{R}^N)$, $\rho_\varepsilon(\mathbf{y}) := \frac{1}{\varepsilon^N}\rho(\frac{\mathbf{y}}{\varepsilon})$ with $\rho \in C_c^\infty(\mathbb{R}^N), \rho \geq 0, spt(\rho) \subset \{|\mathbf{y}| \leq 1\}, \|\rho\|_{1;\mathbb{R}^N} = 1$

- For each $\varepsilon > 0$, $u_\varepsilon \in C^\infty(\mathbb{R}^N)$ *and* $D^\alpha(\rho_\varepsilon * u) = (D^\alpha \rho_\varepsilon) * u$;
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When $u = \chi_E$ for a set of finite perimeter, E, and u_ε is the mollification of χ_E, we have the following stronger results, besides the above properties:

- There is a set $\mathcal{N}$ with $\mathcal{H}^{N-1}(\mathcal{N}) = 0$ and a function $u_{_E} \in BV$ such that, for all $y \notin \mathcal{N}$, $u_\varepsilon(y) \to u_{_E}(y)$ as $\varepsilon \to 0$ and

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Balance Law: Cauchy Flux (Cauchy 1823-27)

An oriented surface: A pair $(S, \boldsymbol{\nu})$ so that $S \Subset \Omega$ is a Borel set and $\boldsymbol{\nu} : \mathbb{R}^N \to \mathbb{S}^{N-1}$ is a Borel measurable unit vector field that satisfy: $\exists$ a set $E \Subset \Omega$ of finite perimeter such that $S \subset \partial^* E$ and $\boldsymbol{\nu}(y) = \boldsymbol{\nu}_{E}(y)\,\chi_S(y)$, where $\boldsymbol{\nu}_{E}(y)$ is the interior measure-theoretic unit normal to E at y. Two oriented surfaces $(S_j, \boldsymbol{\nu}_j), j = 1, 2$, are said to be compatible if $\exists$ a set of finite perimeter E such that $S_j \subset \partial^* E$ and $\boldsymbol{\nu}_j(y) = \boldsymbol{\nu}_{E}(y)\,\chi_{S_j}(y)$, $j = 1, 2$.

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Let Ω be a bounded open set. A Cauchy flux is a functional $\mathcal{F}$ that assigns to each oriented surface $S := (S, \boldsymbol{\nu}) \Subset \Omega$ a real number and satisfies:

(i) $\mathcal{F}(S_1 \cup S_2) = \mathcal{F}(S_1) + \mathcal{F}(S_2)$ for any pair of compatible disjoint surfaces $S_1, S_2 \Subset \Omega$;

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Connection: Cauchy Fluxes and $\mathcal{DM}$-Fields

Let $\mathcal{F}$ be a Cauchy flux in $\mathcal{D}$.

$\Rightarrow$ There exists a unique $\mathcal{DM}$-field $\mathbf{F} \in \mathcal{DM}^{\infty}_{loc}(\mathcal{D})$ such that, for any $(S, \boldsymbol{\nu}) \Subset \mathcal{D}$ with $\sigma(S) = 0$,

$$\mathcal{F}(S) = -\int_S \mathbf{F} \cdot \boldsymbol{\nu} \, d\mathcal{H}^{N-1}, \qquad (\mathbf{F} \cdot \boldsymbol{\nu})|_S = -(\mathbf{F} \cdot (-\boldsymbol{\nu}))|_S,$$

where $\mathbf{F} \cdot \boldsymbol{\nu}$ is the normal trace of $\mathbf{F}$ to the oriented surface.

In particular, when $S = \partial E$, $\mathcal{F}(\partial E) = \int_E \operatorname{div} \mathbf{F}$.

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?? Recovery of the Cauchy flux on S (shock wave) with $\sigma(S) > 0$??

?? Definition of the normal traces on S: $(\mathbf{F} \cdot (\pm\boldsymbol{\nu}))|_S$??

Then we can define the Cauchy flux on S:

$$\mathcal{F}(\pm S) := -\int_{\pm S} \mathbf{F} \cdot (\pm\boldsymbol{\nu}) \, d\mathcal{H}^{N-1}.$$

In general, $\mathcal{F}(S) \neq -\mathcal{F}(-S)$.

Balance Laws (cf. Dafermos's Book 2005, 2nd Edition)

Production: A functional $\mathcal{P}$, defined on any bounded measurable subset of finite perimeter, $E \subset \mathcal{D} \subset \mathbb{R}^N$, taking value in $\mathbb{R}^k$ and satisfying the conditions:

$$|\mathcal{P}(E)| \leq \sigma(E),$$
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Balance Law: A balance law on an open subset $\Omega \subset \mathcal{D}$ postulates that the **production** $\mathcal{P}$ of a vector-valued "extensive" quantity in any bounded measurable subset $E \Subset \Omega$ with finite perimeter is balanced by the **Cauchy flux** $\mathcal{F}$ of this quantity through the reduced boundary $\partial^* E$ of E:

$$\mathcal{P}(E) = \mathcal{F}(\partial^* E).$$

Balance Laws–Conti.

Fugele's theorem $\Rightarrow$ $\exists\ P \in \mathcal{M}(\mathcal{D}; \mathbb{R}^k)$ such that

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Constitutive equations: $\mathbf{F}(y) := \mathbf{F}(\mathbf{u}(\mathbf{y}), \mathbf{y})$ and $P(\mathbf{y}) := P(\mathbf{u}(\mathbf{y}), \mathbf{y})$.

$\Longrightarrow$ **Systems of Balance Laws**: $\quad \mathrm{div}\,\mathbf{F}(\mathbf{u}(\mathbf{y}), \mathbf{y}) = P(\mathbf{u}(\mathbf{y}), \mathbf{y}).$

Conservation Laws: $\mathcal{P} = 0$

The previous derivation yields

$$\operatorname{div} \mathbf{F}(\mathbf{u}(\mathbf{y}), \mathbf{y}) = 0,$$

which is called a system of conservation laws. When the medium is homogeneous: $\mathbf{F}(\mathbf{u}, \mathbf{y}) = \mathbf{F}(\mathbf{u})$, that is, $\mathbf{F}$ depends on $\mathbf{y}$ only through the state vector, then

$$\operatorname{div} \mathbf{F}(\mathbf{u}(\mathbf{y})) = 0.$$

In particular, when the coordinate system $\mathbf{y}$ is described by the time variable t and the space variable $\mathbf{x} = (x_1, \cdots, x_d)$:

$$\mathbf{y} = (t, x_1, \cdots, x_d) = (t, \mathbf{x}), \quad N = d + 1,$$

and the flux density is written as

$$\mathbf{F}(u) = (\mathbf{u}, f_1(\mathbf{u}), \cdots, f_d(\mathbf{u})) = (\mathbf{u}, \mathbf{f}(\mathbf{u})),$$

then we have the following standard form for the **system of conservation laws**:

$$\partial_t \mathbf{u} + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0, \qquad \mathbf{x} \in \mathbb{R}^d, \ \mathbf{u} \in \mathbb{R}^m.$$

Divergence-Measure Fields: $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$

- $\exists$ C^∞ vector fields $\mathbf{F}^j, j = 1, 2, \ldots,$ such that
$$|\mathrm{div}\, \mathbf{F}^j|(\mathcal{D}) \to |\mathrm{div}\, \mathbf{F}|(\mathcal{D});$$

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- If $g \in BV(\mathcal{D})$, then $g\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$ and the product rule holds:

$$\mathrm{div}\,(g\mathbf{F}) = g^*\mathrm{div}\,\mathbf{F} + \overline{\mathbf{F} \cdot \nabla g},$$

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- If $E \Subset \mathcal{D}$ is a set of finite perimeter with $|\mathrm{div}\,\mathbf{F}|(\partial E) = 0$, then the Gauss-Green formula holds on E:
$$\int_E \mathrm{div}\,\mathbf{F} = -\int_{\partial^* E} \mathbf{F}(\mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y})\, d\mathcal{H}^{N-1}(\mathbf{y}).$$

See: Chen-Frid: ARMA 1999

Related: Fuglede, Anzellotti, Ziemer, $\cdots\cdots$

Deformable Lipschitz Boundary $\partial\Omega$: $\Omega \subset \mathbb{R}^N$ open, bdd

- For any $\mathbf{x} \in \partial\Omega$, there exist $r > 0$ and a Lipschitz map $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabeling coordinates if necessary,

$$\Omega \cap Q(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^N \ : \ \gamma(y_1, \dots, y_{N-1}) < y_N \} \cap Q(\mathbf{x}, r),$$

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- There exists $\Psi : \partial\Omega \times [0, 1] \to \overline{\Omega}$ such that Ψ is a homeomorphism, bi-Lipschitz over its image, and $\Psi(\omega, 0) = \omega$ for any $\omega \in \partial\Omega$. The map Ψ is called a Lipschitz deformation of the boundary $\partial\Omega$.

Notations: For $s \in [0, 1]$,

$\partial\Omega_s := \Psi(\partial\Omega \times \{s\})$

Ω_s=the open subset of Ω bounded by $\partial\Omega_s$.

Theorem (C-Frid 1999): ∂E Lip. Deformable; $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$

There exists $\mathbf{F} \cdot \nu \in L^\infty(\partial\Omega)$ such that, for any $\varphi \in C_0^1(\mathbb{R}^N)$,

- $\int_E \varphi \, \mathrm{div}\, \mathbf{F} = -\int_{\partial E} \varphi \, (\mathbf{F} \cdot \nu) \, d\mathcal{H}^{N-1} - \int_E \mathbf{F} \cdot \nabla\varphi \, d\mathbf{y}$;

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- Let $h: \mathbb{R}^N \to \mathbb{R}$ be the level set function of $\partial\Omega_s$:

$$h(\mathbf{y}) := \begin{cases} 0 & \text{for } \ \mathbf{y} \in \mathbb{R}^N - \bar{\Omega}, \\ 1 & \text{for } \ \mathbf{y} \in \Omega - \Psi(\partial\Omega \times [0,1]), \\ s & \text{for } \ \mathbf{y} \in \partial\Omega_s, 0 \le s \le 1. \end{cases}$$

Then, for any $\psi \in \mathrm{Lip}(\partial\Omega)$,

$$\langle \mathbf{F}\cdot\nu,\, \psi\rangle_{\partial\Omega} = \lim_{s\to 0} \frac{1}{s} \int_{\Psi(\partial\Omega\times(0,s))} \mathcal{E}(\psi)\, \nabla h \cdot \mathbf{F}\, d\mathbf{y},$$

where $\mathcal{E}(\psi)$ is any Lipschitz extension of ψ to the whole space $\mathbb{R}^N$.

Approximation Theorem: Almost One-Sided Smooth Approximation of a Set of Finite Perimeter, E

The set E cannot be approximated by smooth sets that lie completely in the interior of E.

For example, let U be the open unit disk with a single radius removed. Then $\mathcal{H}^1(\partial U) = 2\pi + 1$, while $\mathcal{H}^1(\partial^* U) = 2\pi$. Thus, if U_k is an approximating open subset of U, then its boundary will be close to that of boundary U and so $\mathcal{H}^1(\partial U_k)$ will be close to $2\pi + 1$. Adding more radii, say m of them, will force the approximating set to have boundaries whose Hausdorff measure close to 2π plus m.

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Theorem (Chen-Torres-Ziemer 2007)

Let μ be a Radon measure on $\mathbb{R}^N$ satisfying $|\mu| \ll \mathcal{H}^{N-1}$. Then there exists a family of smooth sets A_k such that

- $|\mu|\big((A_k \setminus E^1) \cup (E^1 \setminus A_k)\big) \to 0$;
- $\lim_{k\to\infty} \mathcal{H}^{N-1}(\partial A_k \cap (E^0 \cup \partial^* E)) = 0$.

Theorem (Chen-Torres-Ziemer 2007): $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$

$E := E^1 \cup \partial^* E \Subset \mathcal{D}$ be a bounded set of finite perimeter

There exists a signed measure σ_{in} and a family of sets A_k with smooth boundaries such that

- $\|\mathrm{div}\,\mathbf{F}\|\big((A_k \setminus E^1) \cup (E^1 \setminus A_k)\big) \to 0$;

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- $\sigma_{in} := w^* - \lim_{k\to\infty} \sigma_k$ in $\mathcal{M}(\mathcal{D})$ is carried by $\partial^* E$ with $\|\sigma_{in}\| \, (\mathcal{D} \setminus \partial^* E) = 0$ and $\|\sigma_{in}\| \ll \mathcal{H}^{N-1} \llcorner \partial^* E$;

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- The density of σ_{in}, $(\mathbf{F} \cdot \boldsymbol{\nu})_{in}$, is called the interior normal trace relative to E of $\mathbf{F}$ on $\partial^* E$ and satisfies the Gauss-Green formula:

$$\int_{E^1} \text{div } \mathbf{F} = (\text{div } \mathbf{F})(E^1) = -\sigma_{in}(\partial^* E) = -\int_{\partial^* E} (\mathbf{F} \cdot \nu)_{in}(\mathbf{y}) \, d\mathcal{H}^{N-1}(\mathbf{y});$$

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- $\|\sigma_{in}\| = \|(\mathbf{F}\cdot\nu)_{in}\|_{L^\infty(\partial^* E, \mathcal{H}^{N-1})} \le \|\mathbf{F}\|_{L^\infty(\mathcal{D})}$.

Remark 1

- Let $E \Subset \mathcal{D}$ be a set of finite perimeter. Then, for any $\varphi \in C_0^1(\mathcal{D})$,

$$\int_{E^1} \varphi \operatorname{div} \mathbf{F} = -\int_{\partial^* E} \varphi\, (\mathbf{F} \cdot \boldsymbol{\nu})_i \, d\mathcal{H}^{N-1} - \int_{E^1} \mathbf{F} \cdot \nabla\varphi \, d\mathbf{y}$$

Chen-Torres: ARMA 2005

Šilhavý: Rend. Sem. Mat. Padova, 2005

Remark 2

- Let E is a Lipschitz deformable set. Then there exists a Lipschitz deformation $\{\Psi_\tau\}_{0\le\tau<1}$ such that

$$\partial E_\tau := \Psi_\tau(\partial E) \quad \text{lies completely in the interior of } E$$

and plays the same role of ∂A_k in Chen-Frid: ARMA, 1999.

- More generally, let $E \Subset \mathcal{D}$ is a set of finite perimeter with the property that, for all $\mathbf{y} \in \partial E$, there are positive constants c_0 and r_0 such that

$$\frac{|E^0 \cap B(\mathbf{y}, r)|}{|B(\mathbf{y}, r)|} \ge c_0 \qquad \text{for all } r \le r_0. \qquad\qquad (*)$$

Then there exists $t \in (0,1)$ such that $A_k \Subset E^1$ for large k.

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Condition (*) is similar to Lewis's uniformly flat condition in potential theory (Lewis 1988).

Exterior Normal Traces: $\mathbf{F} \in \mathcal{DM}^\infty(\mathcal{D})$

Let $E := E^1 \cup \partial^* E \Subset \mathcal{D}$ be a bounded set of finite perimeter. Then there exists a signed measure σ_{ex} and a family of sets B_k with smooth boundaries such that

- $\|\text{div } \mathbf{F}\|((B_k \setminus E^1) \cup (E^1 \setminus B_k)) \to 0$;
- $\lim_{k\to\infty} \mathcal{H}^{N-1}(\partial B_k \cap E) = 0$;
- $\lim_{k\to\infty} \|\sigma_k\|(\mathcal{D} \setminus E^0) = \lim_{k\to\infty} \|\sigma_k\|(E) = 0$, where the measures σ_k are defined by $\sigma_k(B) = \int_{B\cap\partial B_k} \mathbf{F} \cdot \nu \, d\mathcal{H}^{N-1}$ for any Borel set $B \subset \mathcal{D}$ with $\mathbf{F} \cdot \boldsymbol{\nu}$ being the normal trace over the smooth boundary ∂B_k;
- $\sigma_{ex} := w^* - \lim_{k\to\infty} \sigma_k$ in $\mathcal{M}(\mathcal{D})$ is carried by $\partial^* E$ with $\|\sigma_{ex}\| (\mathcal{D} \setminus \partial^* E) = 0$ and $\|\sigma_{ex}\| \ll \mathcal{H}^{N-1} \llcorner \partial^* E$;
- The density of σ_{ex}, $(\mathbf{F} \cdot \nu)_{ex}$, is called the exterior normal trace relative to E of $\mathbf{F}$ on $\partial^* E$ and satisfies
$$\int_E \text{div } \mathbf{F} = (\text{div } \mathbf{F})(E) = -\sigma_{ex}(\partial^* E) = -\int_{\partial^* E} (\mathbf{F} \cdot \nu)_{ex}(\mathbf{y}) \, d\mathcal{H}^{N-1}(\mathbf{y});$$
- $\|\sigma_{ex}\| = \|(\mathbf{F} \cdot \nu)_{ex}\|_{L^\infty(\partial^* E, \mathcal{H}^{N-1})} \leq \|\mathbf{F}\|_{L^\infty(\mathcal{D})}$.

Let $\mathcal{F}$ be a Cauchy flux in $\mathcal{D}$.

$\Rightarrow$ There exists a unique $\mathcal{DM}$-field $\mathbf{F} \in \mathcal{DM}^{\infty}_{loc}(\mathcal{D})$ such that, for any oriented surface $\pm S := (S, \pm\boldsymbol{\nu}) \Subset \mathcal{D}$,

$$\mathcal{F}(S) = -\int_S (\mathbf{F}\cdot\boldsymbol{\nu})_{in}\, d\mathcal{H}^{N-1},$$

$$\mathcal{F}(-S) = -\int_{-S} (\mathbf{F}\cdot(-\boldsymbol{\nu}))_{ex}\, d\mathcal{H}^{N-1} = \int_S (\mathbf{F}\cdot\boldsymbol{\nu})_{ex}\, d\mathcal{H}^{N-1}.$$

Connection: Cauchy Fluxes and $\mathcal{DM}$-Fields—Conti.

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$$\Longrightarrow \qquad (\mathrm{div}\,\mathbf{F})(S) = \int_S \left((\mathbf{F}\cdot\boldsymbol{\nu})_{in} - (\mathbf{F}\cdot\boldsymbol{\nu})_{ex}\right) d\mathcal{H}^{N-1}.$$

Physical Principle of Balance Law

$\Longleftrightarrow$ Nonlinear Systems of Balance Laws

$\overset{\mathcal{P}=0}{\Longleftrightarrow}$ Nonlinear Systems of Conservation Laws

- $\mathbf{u}(t,\mathbf{x}) \in \mathcal{M}(\mathbb{R}^{d+1}_+)$ **or** $L^p(\mathbb{R}^{d+1}_+), 1 \le p \le \infty$;
- **For any convex entropy-entropy flux pair** $(\eta, \mathbf{q})$ **so that** $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))(t,\mathbf{x})$ **is a distributional field,**

$$\mu_\eta := \partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) \le 0$$

in the sense of distributions (i.e. $(\eta, \mathbf{q}) := (\eta, q_1, \dots, q_d)$ **is a solution of** $\nabla q_k(\mathbf{u}) = \nabla\eta(\mathbf{u})\nabla\mathbf{f}_k(\mathbf{u}), 1 \le k \le d$**).**

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$\Rightarrow$ For any C^2 entropy pair $(\eta, \mathbf{q})$,

$$(\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x}))) \in \mathcal{DM}^\infty(\mathbb{R}_+ \times \mathbb{R}^n).$$

Cauchy Entropy Flux with respect to η: On any oriented surface S,

$$\mathcal{F}_\eta(S) := \int_S (\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})) \cdot \boldsymbol{\nu} \, d\mathcal{H}^n$$

$(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u})) \cdot \boldsymbol{\nu}$ —The normal trace introduced in CTZ's theorem.

Chen-Frid 2003: $\mathbf{F} \in \mathcal{DM}^p(\mathcal{D})$ for $1 < p < \infty$

Let $E \subset \mathcal{D}$ be a bounded open set with Lipschitz deformable boundary.

- $\exists$ a continuous linear functional $\mathbf{F} \cdot \nu$ over $\mathrm{Lip}(\partial E)$ such that

$$\langle \mathbf{F} \cdot \nu, \phi \rangle_{\partial E} = -\langle \mathrm{div}\, \mathbf{F}, \phi \rangle_E - \int_E \nabla\phi \cdot \mathbf{F}\, dy \qquad \forall\ \phi \in \mathrm{Lip}(\mathbb{R}^N);$$

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- Let $h : \mathbb{R}^N \to \mathbb{R}$ be the level set function of ∂E_s:

$$h(\mathbf{y}) := \begin{cases} 0 & \text{for } \ \mathbf{y} \in \mathbb{R}^N \setminus \bar{E}, \\ 1 & \text{for } \ \mathbf{y} \in E \setminus \Psi(\partial E \times [0,1]), \\ s & \text{for } \ \mathbf{y} \in \partial E_s, 0 \le s \le 1. \end{cases}$$

  Then, $\forall\, \psi \in \mathrm{Lip}(\partial E)$,

$$\langle \mathbf{F} \cdot \nu,\, \psi \rangle_{\partial E} = \lim_{s \to 0} \frac{1}{s} \int_{\Psi(\partial E \times (0,s))} \mathcal{E}(\psi)\, \nabla h \cdot \mathbf{F}\, d\mathbf{y},$$

  where $\mathcal{E}(\psi)$ is any Lipschitz extension of ψ to the whole space $\mathbb{R}^N$;

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$$\langle \mathbf{F} \cdot \nu,\, \psi \rangle_{\partial E} = \lim_{s \to 0} \frac{1}{s} \int_{\Psi(\partial E \times (0,s))} \mathcal{E}(\psi)\, \nabla h \cdot \mathbf{F}\, d\mathbf{y},$$

  where $\mathcal{E}(\psi)$ is any Lipschitz extension of ψ to the whole space $\mathbb{R}^N$;

- $\mathbf{F} \cdot \nu$ can be extended to a continuous linear functional over $W^{1-1/p,p} \cap L^\infty(\partial E)$.

$\mathbf{F} \in \mathcal{DM}^p(\mathcal{D})$ for $1 \le p \le \infty$

- $\|\text{div}\, \mathbf{F}\| \ll \gamma_q$ for $q := p/(p-1)$;
- In view of the fact that φ is defined $\mathcal{H}^{N-q}$-a.e. and therefore μ-a.e., with $\varphi \in W^{1,q}(\mathbb{R}^N)$, then the integral

$$\int_{\mathbb{R}^N} \varphi\, d\,\mu \quad \text{is defined and is meaningful for } \mu := \text{div}\, \mathbf{F}.$$

Definition. For $1 \le q \le N$, the q-**capacity** of an arbitrary set $A \Subset \mathbb{R}^N$ is:

$$\gamma_q(A) := \inf \Big\{ \int_\Omega |\nabla \varphi|^q\, dy \Big\},$$

where the infimum is taken over all test functions $\varphi \in C_c^\infty(\Omega)$ that are identically one in a neighborhood of A.

- $\gamma_q(A) = 0$ for $1 < q < N$ implies that $\mathcal{H}^{N-q+\varepsilon}(A) = 0$ for each $\varepsilon > 0$ and that, conversely, if $\mathcal{H}^{N-q}(A) < \infty$, then $\gamma_q(A) = 0$;
- $\gamma_1(A) = 0$ if and only if $\mathcal{H}^{N-1}(A) = 0$.

Chen-Frid 2003: $\mathbf{F} \in \mathcal{DM}^1(\mathcal{D})$ or $\mathcal{DM}^{ext}(\mathcal{D})$

Let $E \subset \mathcal{D}$ be a bounded open set with Lipschitz deformable boundary.

- $\exists$ a continuous linear functional $\mathbf{F} \cdot \boldsymbol{\nu}$ over $\mathrm{Lip}(\gamma, \partial E)$ for any $\gamma > 1$ such that, for any $\phi \in \mathrm{Lip}(\gamma, \mathbb{R}^N)$,

$$\langle \mathbf{F} \cdot \boldsymbol{\nu},\, \phi \rangle_{\partial E} = -\langle \mathrm{div}\, \mathbf{F},\, \phi \rangle_E - \langle \mathbf{F},\, \nabla \phi \rangle_E.$$

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$$\langle \mathbf{F} \cdot \boldsymbol{\nu},\, \phi \rangle_{\partial E} = -\langle \mathrm{div}\, \mathbf{F},\, \phi \rangle_E - \langle \mathbf{F},\, \nabla \phi \rangle_E.$$

- Let $h : \mathbb{R}^N \to \mathbb{R}$ be the level set function as defined earlier; and, in the case that $\mathbf{F} \in \mathcal{DM}^{ext}(\mathcal{D})$, assume also that $\partial_{x_i} h$ is $|F_i|$-measurable and its set of non-Lebesgue points has $|F_i|$-measure zero, $i = 1, \ldots, N$. Then, for any $\psi \in \mathrm{Lip}(\gamma, \partial E)$, $\gamma > 1$,

$$\langle \mathbf{F} \cdot \boldsymbol{\nu},\, \psi \rangle_{\partial E} = \lim_{s \to 0} \frac{1}{s} \langle \mathbf{F},\, \mathcal{E}(\psi)\, \nabla h \rangle_{\Psi(\partial E \times (0,s))}, \qquad (3)$$

where $\mathcal{E}(\psi) \in \mathrm{Lip}(\gamma, \mathbb{R}^N)$ is the Whitney extension of ψ on ∂E to $\mathbb{R}^N$.

Remark: For this case, the normal trace $\mathbf{F} \cdot \nu$ may no longer be a function on ∂E in general; that is, it cannot be represented as an integrable function w.r.t. the $(N-1)$-dimensional Hausdorff measure over ∂E.

Space $\mathrm{Lip}(\gamma, C)$ on a Closed Set $C \subset \mathbb{R}^N$

Let k be a nonnegative integer and $\gamma \in (k, k+1]$.
We say that a function f, defined on C, belongs to $\mathrm{Lip}(\gamma, C)$ if there exist functions $f^{(j)}$, $0 \le |j| \le k$, defined on C, with $f^{(0)} = f$ so that, if

$$f^{(j)}(x) = \sum_{|j+l| \le k} \frac{f^{(j+l)}(y)}{l!}(x-y)^l + R_j(x,y),$$

then

$$(*) \qquad \begin{cases} |f^{(j)}(x)| \le M, \\ |R_j(x,y)| \le M|x-y|^{\gamma-|j|}, \quad \text{for any } x, y \in C, \ |j| \le k. \end{cases}$$

Here $j = (j_1, \cdots, j_N)$ and $l = (l_1, \cdots, l_N)$ with $j! = j_1! \cdots j_N!$, $|j| = j_1 + j_2 + \cdots + j_N$, and $x^l = x_1^{l_1} x_2^{l_2} \cdots x_N^{l_N}$.
By an element of $\mathrm{Lip}(\gamma, C)$ we mean the collection $\{f^{(j)}(x)\}_{|j| \le k}$. The norm of an element in $\mathrm{Lip}(\gamma, C)$ is defined as the smallest M for which the inequality (*) holds. We notice that $\mathrm{Lip}(\gamma, C)$ with this norm is a Banach space. For the case $C = \mathbb{R}^N$, since the functions $f^{(j)}$ are determined by $f^{(0)}$, this collection is then identified with $f^{(0)}$.

A Different Point of View

Question: Given a Radon measure μ,

?? **find** a continuous or $\mathcal{DM}^p$ vector field that solves the equation:

$$\text{div } \mathbf{F} = \mu \qquad \text{in } \Omega.$$

Existence of a Continuous F:

Bourgain-Brezis (JAMS 2002): $d\mu = f\,dx$ with $f \in L^N_{loc}(\Omega)$

De Pauw-Pfeffer (CPAM 2008): Necessary and Sufficent Condition

μ is a strong charge; i.e., given $\varepsilon > 0$ and a compact set $K \subset \Omega$, there is $\theta > 0$ such that

$$\int_\Omega \phi\, d\mu \le \varepsilon \|\nabla \phi\|_{L^1} + \theta \|\phi\|_{L^1}$$

for any smooth function ϕ compactly supported on K.

Existence of an $\mathbf{F} \in \mathcal{DM}^p, 1 \le p \le \infty$: Phuc-Toress (IUMJ 2008)

Existence of $\mathbf{F} \in \mathcal{DM}^\infty$ if and only if

$\mu(U) \ll C\mathcal{H}^{N-1}(\partial U)$ for any open or closed set with smooth ∂U.

Other Applications in Conservation Laws and Related PDE Problems: Solutions without Bounded Variation

- **Initial-Boundary Value Problems**
- **Decay of Periodic Solutions**
- **Stability of Riemann Solutions (even involving vacuum states)**
- **Traces of Entropy Solutions on Shock Waves ??**
- BV**-like Structure of Entropy Solutions in** L^∞ **??**
- **Generalized Characteristics ??**
- **Free Boundary Problems**
- $\cdots\cdots$

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