# Non-uniqueness of entropy solutions and the carbuncle phenomenon

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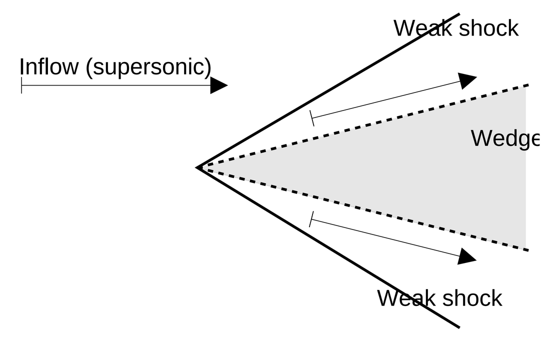
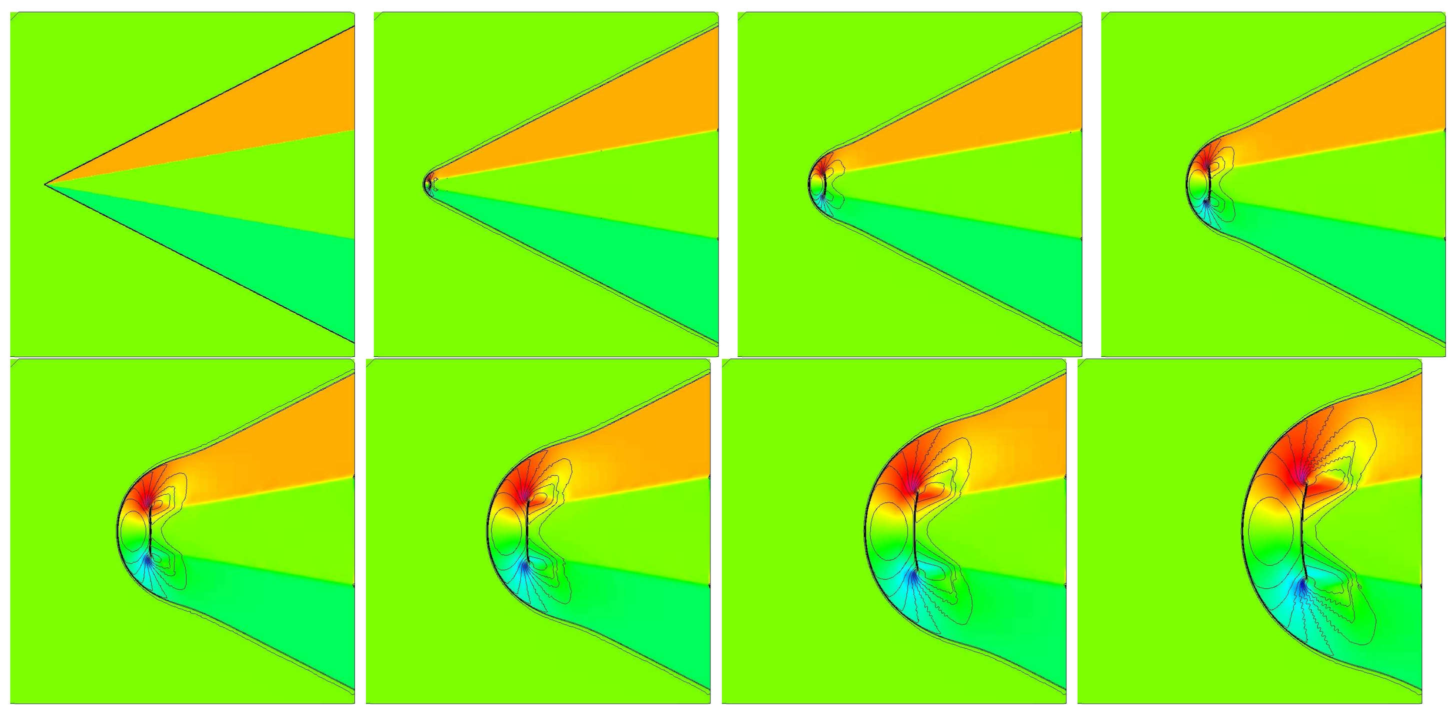
## Numerical non-uniqueness example

Consider supersonic flow onto a solid wedge. Stationary solution with two weak shocks:




This is an entropy solution, steady and self-similar. Now replace the wedge with air of equal pressure as the adjacent air, but zero velocity [Elling, Math. Comp. 2006]. All numerical schemes tested generate a different solution (Mach 3, isentropic Euler, Godunov scheme, colors vertical velocity, contours density):



Essentially the same numerical solution is produced by:

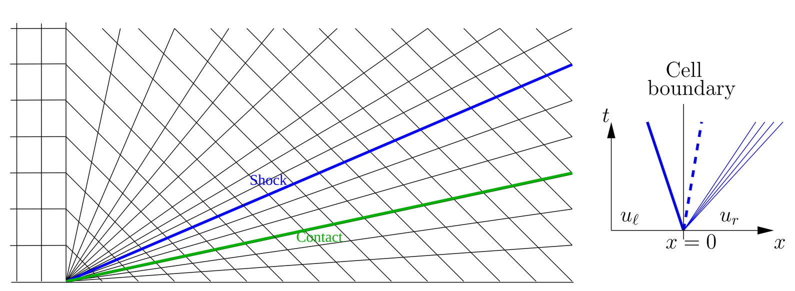
- all numerical schemes tested (Godunov, Roe, Solomon-Osher, ...),
- first-order or higher-order correction (slope limiters),
- uniform Cartesian or adaptive non-Cartesian grids, ...

The non-uniqueness can be observed for

- isentropic or non-isentropic Euler,
- arbitrary coefficient $\gamma$ in the pressure law,
- all Mach numbers, all "wedge" angles,
- both weak and strong reflected shocks.

The numerical solution is self-similar, but unsteady (origin and shock separate by $\approx 350m/s$ in this case). Moreover, all numerical schemes tested yield essentially this solution, with unimportant variations.

## Special grid




On this grid, the initial data is also the numerical solution produced by the Godunov flux (in exact arithmetic). On any other grid, Godunov produces the unsteady solution. Conclusion:

Depending *only* on the grid, the Godunov scheme converges to either different solutions, at least on practical grids. *Convergence theory impossible since false.*

## Interpretation

Theorem [Lax-Wendroff 1960, Elling 2007]:

- If a numerical scheme is *consistent* and *conservative* and if the solutions *converge boundedly a.e.* as the grid is refined, then the limit is a weak solution.
- If the scheme satisfies a discrete entropy inequality as well, then the limit is an entropy solution.

Consequence: since our numerical solutions seem to converge, we expect the second solution to be entropy as well.

**Entropy solutions of the Cauchy problem for the multi-d compressible Euler equations (isentropic or full) are probably not unique.**

[De Lellis/Szekelyhidi 2008]: rigorous (different) counterexample to uniqueness.

## Consequences

For theory:

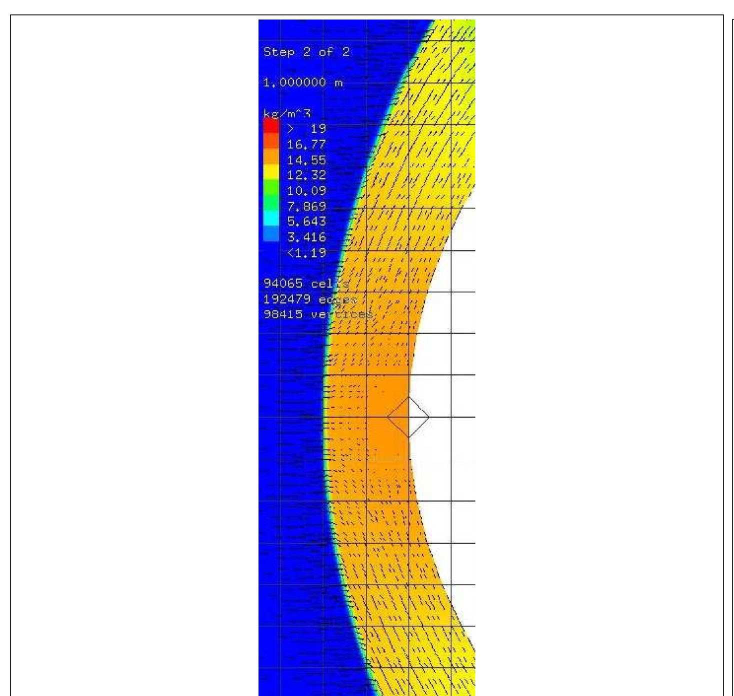
- The entropy inequality is not enough — we need new admissibility criteria.
- Is the vanishing viscosity limit intrinsically ill-posed? [see carbuncles]
- Motivates study of *viscous* genuinely multi-dimensional solutions.

For numerics:

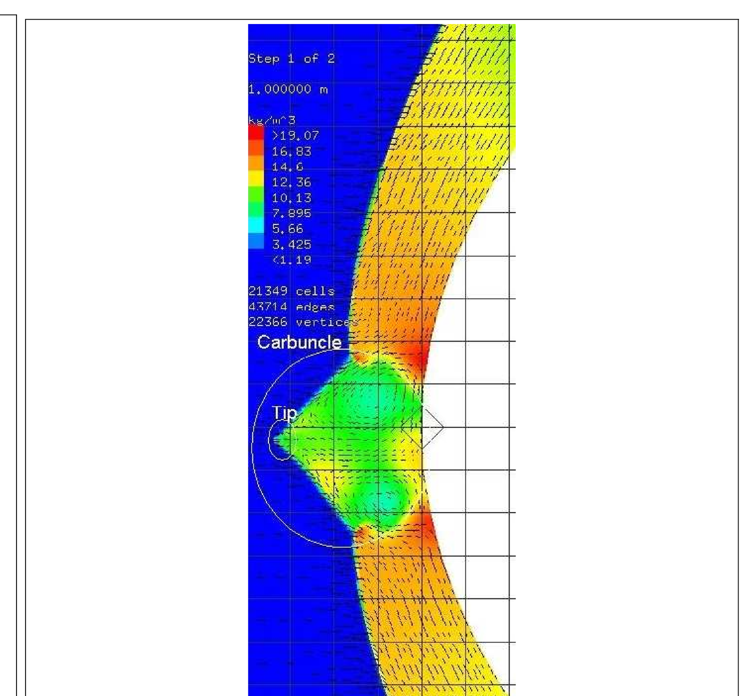
- Discrete entropy inequalities do *not* guarantee convergence to a unique correct solution.
- The Godunov scheme (and perhaps others) does not converge to a unique solution.
- Numerical analysis: interesting convergence theorems may be impossible since false.

## Carbuncles

First observed by Peery/Imlay [1988] in supersonic flow onto blunt bodies (Mach 5 from left):



Lax-Friedrichs: no carbuncle (correct)



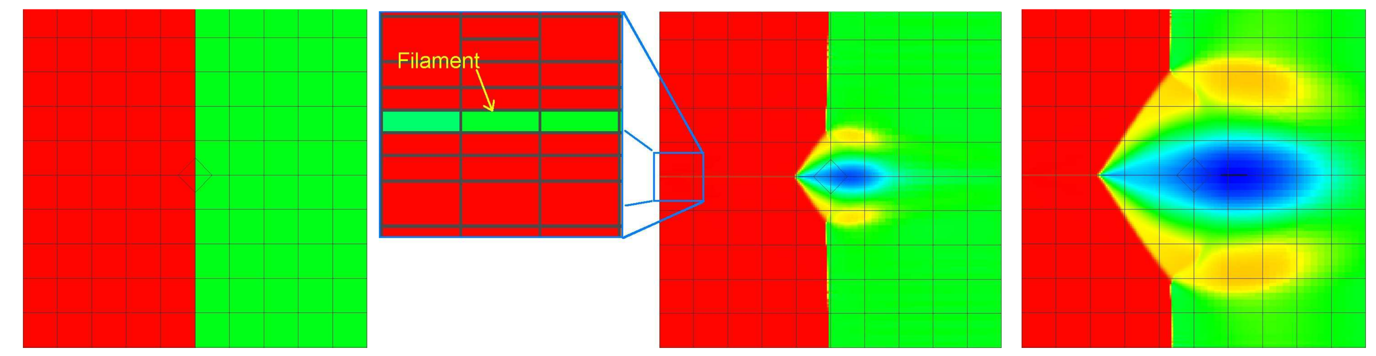

Roe scheme: carbuncle

- Particularly common at high Mach numbers. Folklore: not observed below Mach $\approx 1.5$.
- Carbuncles suppressed by numerical dissipation *near tip*, but robust away from tip.
- Especially susceptible: Godunov scheme, approximate Riemann solvers (Roe, local Lax-Friedrichs, ...).
- Many attempts to *fix* numerical schemes, none completely satisfactory.
- Carbuncles can be absent in one case, present in other for no obvious reason. Absence in a new scheme is never certain. *We need better tests.*

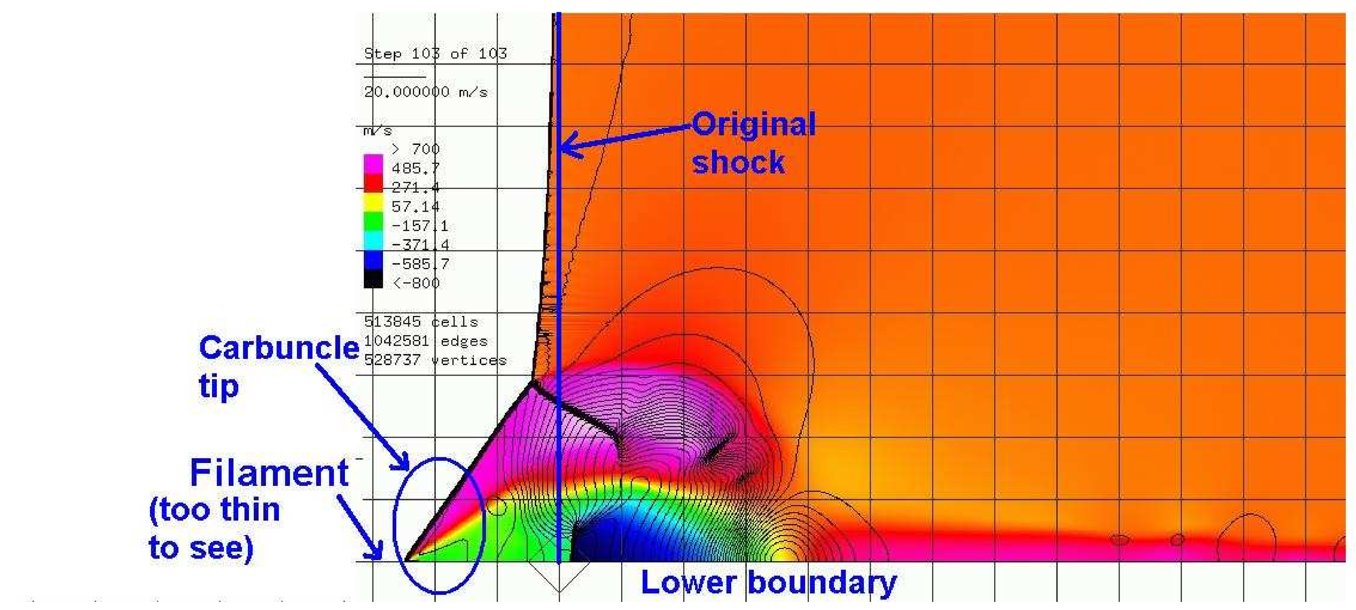
Major obstacle for hypersonic and/or high-resolution flow calculations

## Carbuncles = separating boundary layers

Filament: a (double) vortex sheet generated by setting $\vec{v} = 0$ in one cell at the left (inflow) boundary [Elling 2004].




Result: a carbuncle in an originally straight shock. The flow is clearly self-similar, so repeat the calculation in $(x/t, y/t)$ coordinates [Godunov, Mach 3, isentropuc Euler]:




- Carbuncles are self-similar (unless blunt bodies etc. provide a natural length scale).
- The filament provides a new, ***reliable*** test for carbuncles.
- Observe: the carbuncle tip is the initial data of the non-uniqueness example.
- The filament resembles
  - **boundary layers** (physical or numerical), or
  - wakes of upstream objects.

Conclusions:

The carbuncle phenomenon is another manifestation of non-uniqueness of entropy solutions.

*Both* solutions of the non-uniqueness example may occur in physics.

Conjecture: we can pick any point of a smooth shock and insert a carbuncle of arbitrary tip angle and (perhaps) tip location.

Carbuncles are separating boundary layers (once reflected across the boundary). They are *physically meaningful* solutions, not numerical artefacts.

## Fixing numerical schemes

- Assume the small (viscous) scales are such that there are no carbuncles. *This assumption may be wrong in some applications!*
- P. Roe: try to measure and control the shock curvature without adding dissipation [ongoing research].
- Problem: small curvature assumption fails at
  - shock interactions/reflections,
  - boundary layers, especially if they *do* separate,
  - shock-vortex sheet interactions,
  - shock-entropy wave interactions
  - ...

In many cases, modelling small scales (boundary layer equations, turbulence models, ...) may be necessary: pure Euler schemes *cannot* be physically accurate.