Conservation laws on networks

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joint works with R.M. Colombo and B. Piccoli

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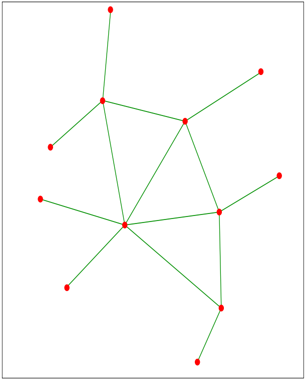
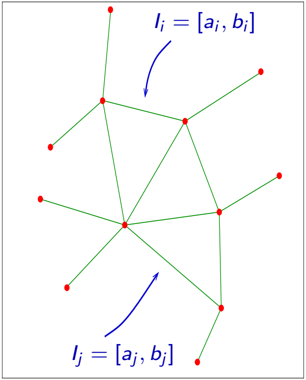


A network is a finite collection of arcs and vertices

Conservation Laws on Networks



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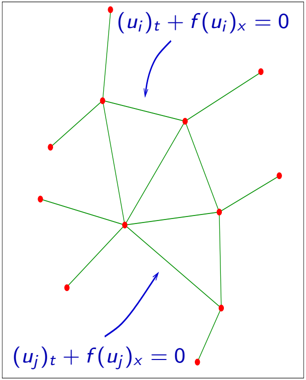


A network is a finite collection of arcs and vertices

Each arc is modeled by $I_i = [a_i, b_i]$

On each arc we consider the PDE system $(u_i)_t + f(u_i)_x = 0$

Various applications

Car traffic

LWR model

$$\rho_t + (\rho v(\rho))_x = 0$$

- Coclite, G., Piccoli, *SIAM J. Math. Anal.* 36, 2005.
- Holden, Risebro, *SIAM J. Math. Anal.* 26, 1995.
- Lighthill, Whitham, *Proc. Roy. Soc. London Ser. A* 229, 1955.
- Richards, *Oper. Res.* 4, 1956.

Various applications

Car traffic

Aw-Rascle-Zhang model

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (v + p(\rho))_t + v(v + p(\rho))_x = 0, \end{cases}$$

- Aw, Rascle, *SIAM J. Appl. Math.* 60, 2000.
- G., Piccoli, *Commun. Partial Differential Equations* 31, 2006.
- Zhang, *Transportation Research B* 36, 2002.

Various applications

Car traffic

Colombo phase transition model

Free flow

$$\begin{cases} (\rho, q) \in \Omega_f, \\ \rho_t + [\rho \cdot v]_x = 0, \\ v = \left(1 - \frac{\rho}{R}\right) \cdot V, \end{cases}$$

Congested flow

$$\begin{cases} (\rho, q) \in \Omega_c, \\ \rho_t + [\rho \cdot v]_x = 0, \\ q_t + [(q - Q) \cdot v]_x = 0, \\ v = \left(1 - \frac{\rho}{R}\right) \cdot \frac{q}{\rho}. \end{cases}$$

- Colombo, *SIAM J. Appl. Math.* 63, 2002.
- Colombo, Goatin, Piccoli, *J. Hyperbolic Differ. Equ.*, 2009.
- Colombo, Goatin, Priuli, *Nonlinear Anal.* 66, 2007.

- Goatin, *Math. Comput. Modelling* 44, 2006.

Various applications

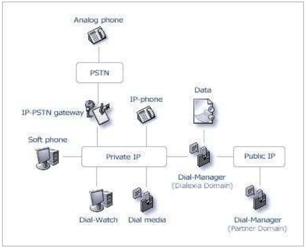
Gas pipelines

p-system

$$\begin{cases} \rho_t + q_x = 0, \\ q_t + \left(\frac{q^2}{\rho} + p(\rho)\right)_x = 0, \end{cases}$$

- Banda, Herty, Klar, *Netw. Heterog. Media* 1, 2006.
- Colombo, G., *SIAM J. Math. Anal.* 39, 2008.
- Colombo, Guerra, Herty, Sachers, *SIAM J. Contol Optim.* 48, 2009.
- Colombo, Herty, Sachers, *SIAM J. Math. Anal.* 40, 2008.
- Colombo, Mauri, *J. Hyperbolic Diff. Eq.* 5, 2008.
- Colombo, Marcellini, *J. Matematical Anal. and Appl.*

Various applications

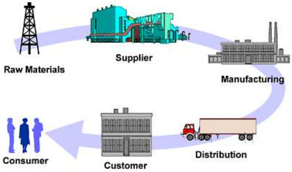


Data networks

$$\rho_t + f(\rho)_x = 0$$

- D'Apice, Manzo, Piccoli, *SIAM J. Appl. Math.* 68, 2008.

Various applications

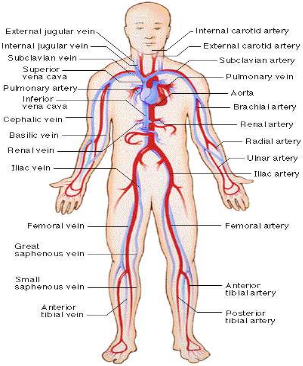


Supply chains

$$\rho_t + (\min\{\mu(t,x), v\rho\})_x = 0$$

- Armbruster, Degond, Ringhofer, *Bull. Inst. Math. Acad. Sin.* 2, 2007.
- Göttlich, Herty, Klar, *Commun. Math. Sci.* 4, 2006.
- D'Apice, Göttlich, Herty, Piccoli, *SIAM book series*, 2009.

Various applications



Blood circulation

$$\begin{cases} A_t + m_x = 0 \\ m_t + \left(\frac{\alpha m^2}{A}\right)_x + \frac{A}{\rho} p_x = -K\frac{m}{A} \end{cases}$$

- Čanić, Kim, *Math. Meth. Appl. Sci.* 26, 2003.
- Fernández, Milišić, Quarteroni, *Multiscale Model. Simul.* 4, 2005.

Various applications

Irrigation channel

De Saint-Venant equation

$$\begin{cases} H_t + (Hv)_x = 0 \\ v_t + \left(\frac{1}{2}v^2 + gH\right)_x = gS(H, v) \end{cases}$$

- Coron, d'Andréa-Novel, Bastin, ECC1999.
- Gugat, Leugering, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26, 2009.

The LWR model

$$\rho_t + f(\rho)_x = 0$$

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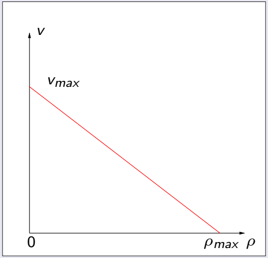
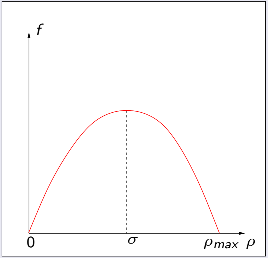
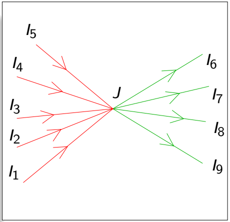




Solutions at nodes

- J node: n incoming arcs, m outgoing arcs



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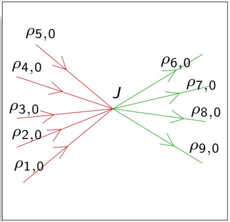
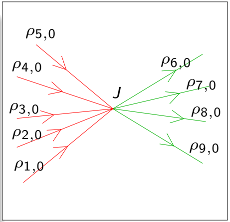


Solutions at nodes

- J node: n incoming arcs, m outgoing arcs
- Consider an initial datum $\rho_{0,l}$ in each arc
- This corresponds to $n+m$ IBV problems

$$\begin{cases} (\rho_l)_t + f(\rho_l)_x = 0, & l \in \{1, \dots, n+m\} \\ \rho_l(0,x) = \rho_{l,0}(x), & x \in I_l, l \in \{1, \dots, n+m\} \\ \rho_l(t,0) = ?, & t > 0, l \in \{1, \dots, n+m\} \end{cases}$$



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Giving a solution is equivalent to giving its trace at the node.

Solutions at nodes ... the Riemann problem

A Riemann problem at a node is a Cauchy problem with constant initial conditions on each arc.

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Giving a solution is equivalent to giving its trace at the node.

Definition

A Riemann solver at the node J is a function

$$\mathcal{RS} : [0, \rho_{max}]^{n+m} \to [0, \rho_{max}]^{n+m},$$

which gives the trace at the node of a solution to the corresponding Riemann problem.

The traces

- Given an arc of a node and an initial datum $\rho_{l,0}$, not all the elements in $[0, \rho_{max}]$ can be the trace at the node of a solution to a Riemann problem

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- We prescribe the conservation at J:

$$\sum_{i=1}^{n} f(\hat{\rho}_i) = \sum_{j=n+1}^{n+m} f(\hat{\rho}_j)$$

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- There are infinitely many Riemann solvers with these properties!

A particular Riemann solver

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$$A \cdot (f(\hat{\rho}_1), \dots, f(\hat{\rho}_n))^T = (f(\hat{\rho}_{n+1}), \dots, f(\hat{\rho}_{n+m}))^T$$

on the fluxes at J

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Remark

If we "invert" the order of the last two rules, i.e. first we maximize the functional and then we impose some constraints, then we obtain a different Riemann solver.

Properties (P1), (P2) and (P3)

Property (P1)

$\mathcal{RS}$ has the property (P1) if, given $(\rho_{1,0}, \dots, \rho_{n+m,0})$ and $(\rho'_{1,0}, \dots, \rho'_{n+m,0})$ such that $\rho_{l,0} = \rho'_{l,0}$ whenever $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad datum, then

$$\mathcal{RS}(\rho_{1,0}, \dots, \rho_{n+m,0}) = \mathcal{RS}(\rho'_{1,0}, \dots, \rho'_{n+m,0}).$$

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The datum $\rho_{l,0}$ is called bad if

$$\rho_{l,0} < \sigma, \quad l \leq n \qquad \text{or} \qquad \rho_{l,0} > \sigma, \quad l \geq n+1$$

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i.e. $\rho_{l,0}$ gives a non trivial constraint for the flux solution at J

Properties (P1), (P2) and (P3)

$\mathcal{RS}$ has the property (P2) if there exists $C \geq 1$ such that, for every equilibrium $(\rho_{1,0}, \dots, \rho_{n+m,0})$ for $\mathcal{RS}$ and for every wave $(\rho_{l,0}, \rho_l)$ ($l \in \{1, \dots, n+m\}$) interacting with J at time $\bar{t}$, it holds

$$\text{Tot.Var.}_f(\bar{t}+) - \text{Tot.Var.}_f(\bar{t}-) \leq C \min\left\{\left|f(\rho_{l,0}) - f(\rho_l)\right|, \left|\Gamma(\bar{t}+) - \Gamma(\bar{t}-)\right|\right\}.$$

- An equilibrium is a fixed point of $\mathcal{RS}$
- The functionals $\Gamma(t)$ and $\text{Tot.Var.}_f(t)$ are defined by

$$\Gamma(t) := \sum_{i=1}^{n} f(\rho_i(t, 0-))$$

$$\text{Tot.Var.}_f(t) := \sum_{l=1}^{n+m} \text{Tot.Var.} f(\rho_l(t, \cdot))$$

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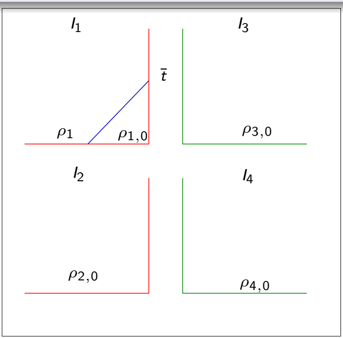

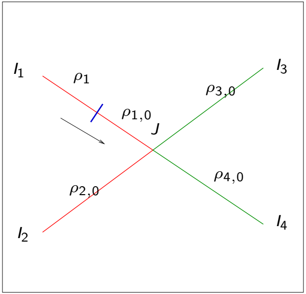

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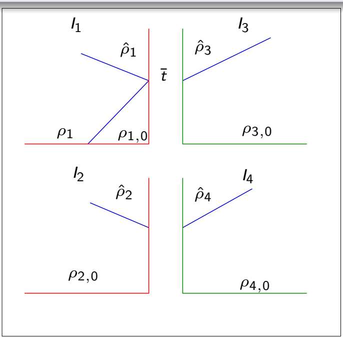

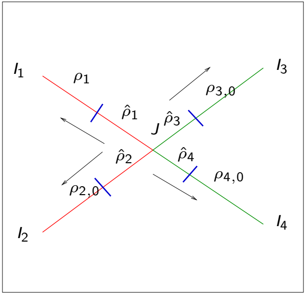

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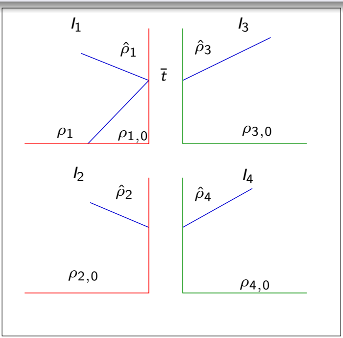
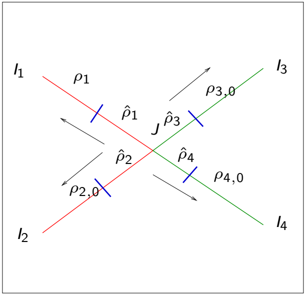
$$|f(\rho_1) - f(\hat\rho_1)| + |f(\rho_{2,0}) - f(\hat\rho_2)| + |f(\rho_{3,0}) - f(\hat\rho_3)| + |f(\rho_{4,0}) - f(\hat\rho_4)| \\ - |f(\rho_{1,0}) - f(\rho_1)| \leq C \min\left\{|f(\rho_{1,0}) - f(\rho_1)|, |f(\hat\rho_1) + f(\hat\rho_2) - f(\rho_{1,0}) - f(\rho_{2,0})|\right\}.$$





Properties (P1), (P2) and (P3)

$\mathcal{RS}$ satisfies the property (P3) if, for every equilibrium $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ of $\mathcal{RS}$ and for every wave $(\rho_{l,0}, \rho_l)$ ($l \in \{1, \ldots, n+m\}$) with $f(\rho_l) < f(\rho_{l,0})$, interacting with J at time $\bar{t} > 0$ it holds

$$\Gamma(\bar{t}+) \leq \Gamma(\bar{t}-).$$

Main result

Theorem [AIP 2009]

Let $\mathcal{RS}$ be a Riemann solver satisfying properties (P1)–(P3). Fix initial conditions $\rho_{l,0} \in BV$.

For every $T > 0$, there exists a solution $(\rho_1, \ldots, \rho_{n+m})$ to the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\rho_l + \frac{\partial}{\partial x} f(\rho_l) = 0 \\ \rho_l(0,x) = \rho_{l,0}(x) \end{cases} \qquad l = 1, \ldots, n+m$$

such that

$$\mathcal{RS}(\rho_1(t,0), \ldots, \rho_{n+m}(t,0)) = (\rho_1(t,0), \ldots, \rho_{n+m}(t,0))$$

for a.e. $t \in [0, T]$.

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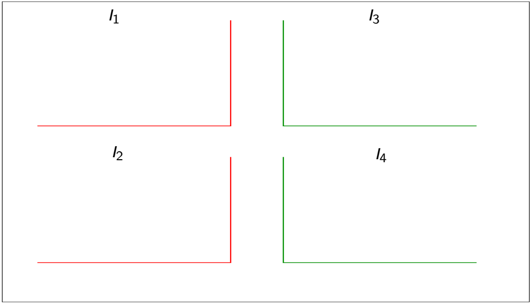
Remark [JDE 2009]

The previous result holds also in the case of Riemann solvers $\mathcal{RS}$ depending on time

proof

- The proof is based on the wave-front tracking method



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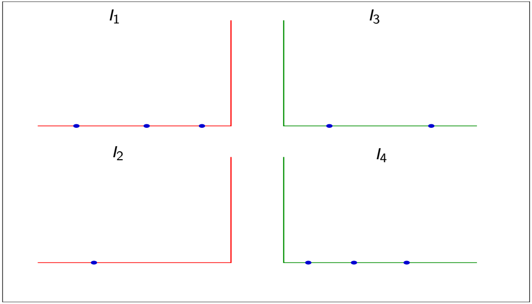

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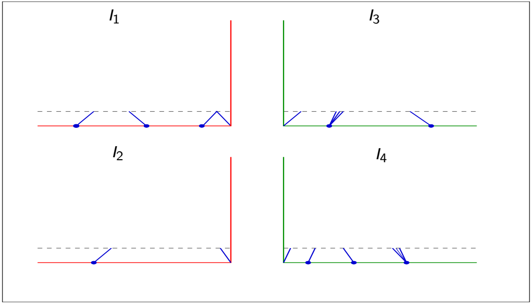

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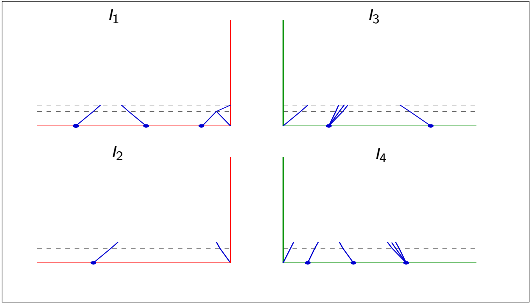

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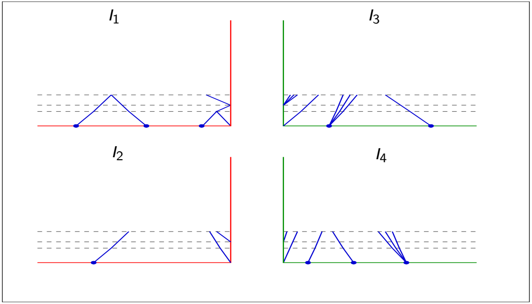

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3. In general, there is not a bound for the total variation of the density

Continuous dependence

Theorem [AIP 2009]

Fix a node J and a Riemann solver $\mathcal{RS}$ satisfying (P1), (P2) and (P3) and such that

$$\mathrm{Tot.Var.}_f(\bar{t}+) \leq \mathrm{Tot.Var.}_f(\bar{t}-)$$

for every time $\bar{t}$ at which an interaction of a wave with J happens. Then there exists a unique solution to the Cauchy problem and the solution depends in a Lipschitz continuous way on the initial datum with respect to the L^1-topology.

Continuous dependence (idea of the proof)

1. Introduce a differential structure on L^1.
[Bressan, Colombo, 1995] [Bressan, Crasta, Piccoli, 2000]
$\gamma : [0,1] \to L^1$ a curve s.t. $\gamma(\theta)$ is a piecewise constant functions with N discontinuities: $x_1(\theta) < x_2(\theta) < \cdots < x_N(\theta)$.
Tangent vector $\dot\gamma(\theta) = (v,\xi)(\theta) \in L^1 \times \mathbb{R}^N$ if

$$L^1 \ni v(\theta,x) \doteq \lim_{h\to 0} \frac{\gamma(\theta+h,x)-\gamma(\theta,x)}{h}, \qquad \text{for a.e. } x,$$

$$\xi_i(\theta) \doteq \lim_{h\to 0} \frac{x_i(\theta+h)-x_i(\theta)}{h}, \qquad i = 1,\dots,N.$$

The norm of $(v,\xi)(\theta)$ is defined by:

$$\|(v,\xi)(\theta)\| = \|v(\theta)\|_{L^1} + \sum_{i=1}^{N} |\xi_i(\theta)| |\gamma(\theta,x_i+) - \gamma(\theta,x_i-)|.$$

Continuous dependence (idea of the proof)

1. Introduce a differential structure on L^1.
2. Define a distance between piecewise constant functions

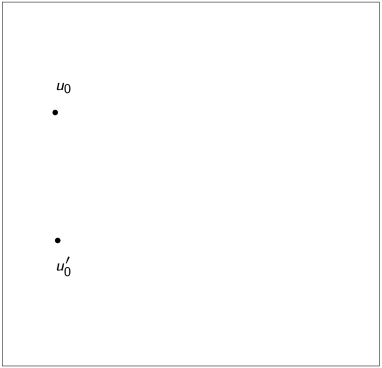
$$d(u, u') = \inf \int_0^1 \|\dot{\gamma}(t)\|$$

$$d(u, u') \sim \left\| u - u' \right\|_{L^1}$$

Continuous dependence (idea of the proof)

1. Introduce a differential structure on L^1.
2. Define a distance between piecewise constant functions
3. Prove that the norm of tangent vectors are not increasing in time along wave front tracking solutions

Continuous dependence (idea of the proof)



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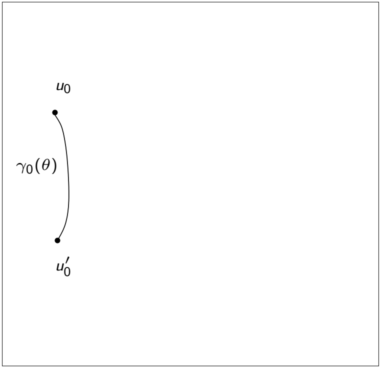

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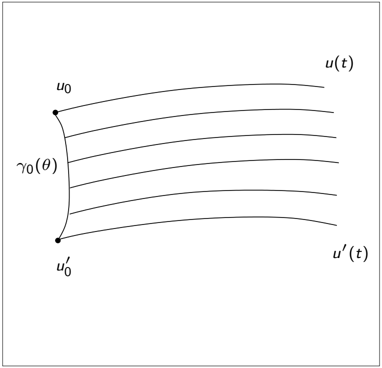

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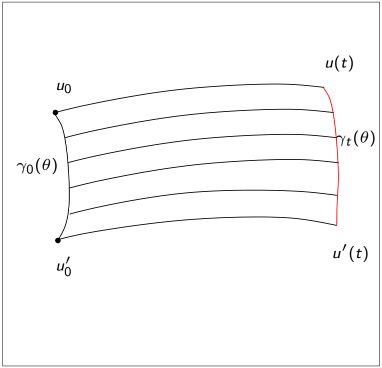

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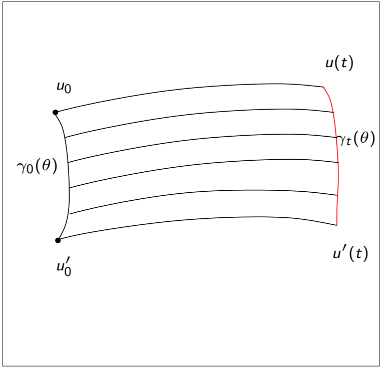

For a.e. θ, the norm $\|\dot{\gamma}_s(\theta)\|$ is not increasing with respect to s.

The p-system in a tube

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho) \right) = 0 \end{cases}$$

- $\rho(t,x)$ density of the fluid

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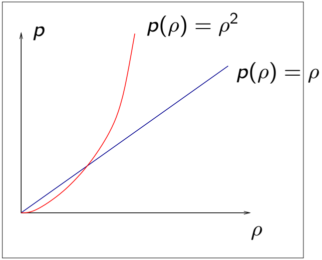
- $p(\rho)$ pressure law

The pressure law p

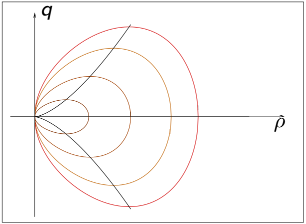
$$p \in C^2\left(]0,+\infty[\,;\,]0,+\infty[\right) \qquad p' > 0 \qquad p'' \geq 0$$

Typical case: $p(\rho) = k\rho^\gamma \qquad \gamma \geq 1$



Lax curves and other quantities

- "dynamic pressure": $P(\rho, q) = \frac{q^2}{\rho} + p(\rho)$



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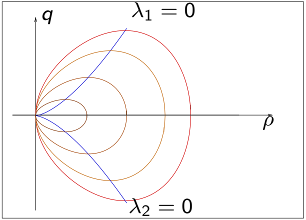

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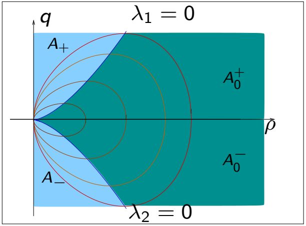

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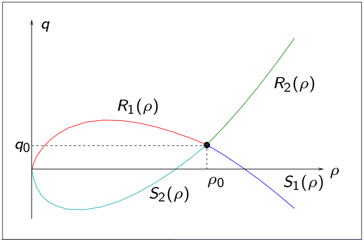
$$\begin{cases} S_1(\rho) = \dfrac{\rho}{\rho_o} q_o - \sqrt{\dfrac{\rho}{\rho_o}(\rho - \rho_o)\left(p(\rho) - p(\rho_o)\right)} & \rho \geq \rho_o \\ S_2(\rho) = \dfrac{\rho}{\rho_o} q_o - \sqrt{\dfrac{\rho}{\rho_o}(\rho - \rho_o)\left(p(\rho) - p(\rho_o)\right)} & \rho \leq \rho_o \\ R_1(\rho) = \dfrac{\rho}{\rho_o} q_o - \rho \displaystyle\int_{\rho_o}^{\rho} \dfrac{\sqrt{p'(r)}}{r}\, dr & \rho \leq \rho_o \\ R_2(\rho) = \dfrac{\rho}{\rho_o} q_o + \rho \displaystyle\int_{\rho_o}^{\rho} \dfrac{\sqrt{p'(r)}}{r}\, dr & \rho \geq \rho_o\,. \end{cases}$$



The Riemann problem at the junction

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(E): $\sum \|\nu_l\| F\left(\rho_l(t,0), q_l(t,0)\right) \leq 0$ (entropy inequality)

$$
\begin{aligned}
E(\rho,q) &= \frac{q^2}{2\rho} + \rho \int_{\rho_*}^{\rho} \frac{p(r)}{r^2}\, dr \\
F(\rho,q) &= \frac{q}{\rho} \cdot \left(E(\rho,q) + p(\rho)\right) \\
\|\nu_l\| &= \text{section of the l-th tube}
\end{aligned}
$$

Stability of solutions for Riemann problems

Theorem [NHM 2006]

Fix n initial states $(\widehat{\rho}_l, \widehat{q}_l) \in \mathring{A}_0$ satisfying

$$\left\{ \begin{array}{l} \sum \|\nu_l\| \, \widehat{q}_l = 0, \\ P(\widehat{\rho}_l, \widehat{q}_l) = P_* \qquad l = 1, \ldots, n \\ \sum \|\nu_l\| \, F(\widehat{\rho}_l, \widehat{q}_l) < 0 \, . \end{array} \right.$$

Then, for every initial datum closed to $(\widehat{\rho}_l, \widehat{q}_l)$, there exists (locally) a unique solution (ρ_l, q_l) to the Riemann problem.

The proof is based on the Implicit Function Theorem

The Cauchy problem

Theorem [SIMA 2008]

Fix n subsonic states $(\hat\rho, \hat q) \in (\mathring{A}_0)$ such that

$$\left\{\begin{array}{l} \sum \|\nu_l\| \, \hat q_l = 0, \\ P(\hat\rho_l, \hat q_l) = P_* \qquad l = 1, \ldots, n \\ \sum \|\nu_l\| \, F(\hat\rho_l, \hat q_l) < 0 \,. \end{array}\right.$$

Then, there exists a map $S\colon [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$, with the properties:

- $\mathcal{D} \supseteq \left\{(\rho, q) \in L^1(\mathbb{R}^+; \mathbb{R}^+ \times \mathbb{R})^n \colon \text{Tot.Var.}(\rho, q) \le \delta_0\right\}$;

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- for $(\rho, q), (\rho', q') \in \mathcal{D}$ and $s, t \geq 0$,

$$\left\| S_t(\rho, q) - S_s(\rho', q') \right\|_{L^1} \leq L \cdot \left(\left\| (\rho, q) - (\rho', q') \right\|_{L^1} + |t - s| \right)\,.$$

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- the map $t \mapsto S_t(\rho, q)$ is a weak entropy solution to the Cauchy Problem.

The Cauchy problem

Interaction estimate at the junction

For any 1-waves σ_l^- hitting the junction and producing the 2-waves σ_l^+, it holds

$$\sum_{l=1}^{n} \left|\sigma_l^+\right| \leq K_J \cdot \sum_{l=1}^{n} \left|\sigma_l^-\right| .$$

The Cauchy problem

Glimm functional

$$V(t) = \sum_{I=1}^{n} \sum_{\alpha \in \mathcal{J}_I} \left[2\, K_J \cdot \left|\sigma_{I,1,\alpha}\right| + \left|\sigma_{I,2,\alpha}\right| + \left|\sigma_{I,3,\alpha}\right|\right]$$

$$Q(t) = \sum_{I=1}^{n} \sum \left\{\left|\sigma_{I,i,\alpha}\, \sigma_{I,j,\beta}\right| : (\sigma_{I,i,\alpha}, \sigma_{I,j,\beta}) \in \mathcal{A}_I\right\}$$

$$\Upsilon(t) = V(t) + K_1 \cdot Q(t)\,,$$

The Cauchy problem

Liu-Yang functional

$$\Phi\left(u_1, u_2\right) = \sum_{l=1}^{n} \sum_{i=1}^{2} \int_0^{+\infty} \left|s_{l,i}(x)\right| \, W_{l,i}(x)\, dx\,,$$

$$W_{l,i}(x) = 1 + \kappa_1\, A_{l,i}(x) + \kappa_1\, \kappa_2\, \left[\Upsilon\left(u_1\right) + \Upsilon\left(u_2\right)\right].$$

- Liu-Yang, Comm. Pure Appl. Math. 52, 1999.
- Bressan-Liu-Yang, Ration. Mech. Anal. 149, 1999.