Optimal Nodal Control Of Networked Systems Of Conservation Laws

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Introduction

Natural gas is transported in pipeline systems over thousands of kilometers and thereby experience a pressure drop due to pipe–wall friction. To compensate these losses controllable compressor stations are introduced. They are supposed to be operated such that customer demands at pipe outlets are satisfied. Similar problems occur in the flow of water in open canals which can be controlled by overflow gates and pumps.

Different approaches have been proposed to treat this problem mathematically. We focus on a 1d description of the flow by a system of nonlinear hyperbolic balance laws. For classical solutions exact controllability and stabilization for different overflow gates in the case of the St. Venant equations has been obtained by Coron et. al., Li Tatsien et. al. and Leugering in recent years for coupled systems.

Below the treatment of weak solutions and results on optimal control for water and gas systems is presented. The results are joint work of R. Colombo (University of Brescia, IT), G. Guerra (University of Milano, IT), M. Herty (RWTH Aachen, GER) and V. Schleper (TU Kaiserslautern, GER).

The Model

The gas dynamics in each pipe are modeled by the p–system

$$\rho_t + q_x = 0$$
$$q_t + \left(\frac{q^2}{\rho} + p(\rho)\right)_x = -\frac{\lambda}{2D}\frac{q|q|}{\rho} - g\rho\sin\alpha \tag{1}$$

with

ρ:	density
q:	flux
λ:	friction factor
D:	pipe diameter
g:	gravitational constant
α:	inclination of the pipe
$p(\rho)$:	pressure law

Often the ideal gas law $p(\rho) = a^2\rho$ is used, where a^2 is the speed of sound in the gas. Furthermore, we restrict to *subsonic states*, i.e. the velocity of the fluid is strictly less than the speed of sound.

At the connection points of the pipes (junctions), we distinguish between *controllable* (compressor) junctions of exactly 2 pipes with compressor power P and *standard* pipe to pipe intersections. To models these junctions, we couple the equations (1) modeling the flow in ach pipe by the following coupling conditions:

1. standard coupling:

$$\sum_{\text{incoming pipes}} q = \sum_{\text{outgoing pipes}} q, \qquad \rho_{\text{in}} = \rho_{\text{out}} \tag{2}$$

for all incoming and outgoing pipes.

2. compresor coupling:

$$q_{\text{in}} = q_{\text{out}}, \qquad P(t) = q_{\text{in}}\left(\left(\frac{\rho_{\text{out}}}{\rho_{\text{in}}}\right)^{\kappa} - 1\right) \tag{3}$$

Existence Results

Junctions w/o control action:
Given a standard junction of $n \geq 2$ ($n \in \mathbb{N}$) pipes with coupling condition (2) and initial conditions $y_0 = \left(\rho_0^{(1)}, q_0^{(1)}, \ldots, \rho_0^{(n)}, q_0^{(n)}\right)$.
The *Cauchy problem* at the junction is then defined by

$$\rho_t^{(i)} + q_x^{(i)} = 0$$
$$q_t^{(i)} + \left(\frac{(q^{(i)})^2}{\rho} + p\left(\rho^{(i)}\right)\right)_x = -\frac{\lambda}{2D}\frac{q^{(i)}|q^{(i)}|}{\rho^{(i)}} - g\rho^{(i)}\sin\alpha$$

in $[0,\infty]$ for each pipe i, with initial condition y_0 and coupling condition (2) at $x = 0$.
Let $\bar{y} := \left(\bar{\rho}^{(1)}, \bar{q}^{(1)}, \ldots, \bar{\rho}^{(n)}, \bar{q}^{(n)}\right)$ be a constant state fulfilling the coupling conditions (2) and $\Omega \subset \mathbb{R}^n$ a non–empty open set. Define

$$\mathcal{D}^\delta = \left\{y \in \bar{y} + L^1\left(\mathbb{R}^+, \Omega\right) : TV(y) < \delta\right\}$$

for some $\delta > 0$.
Then there exist δ, T and a semigroup

$$\mathcal{E} : [0, T - t_0] \times [0, T] \times \mathcal{D}_{t_0} \to \mathcal{D}^\delta$$

for some $\mathcal{D}_{t_0} \subset \mathcal{D}^\delta$ such that $\mathcal{E}(t, t_0, y_0)$ is the solution of the Cauchy problem at the junction with initial condition $y(t_0, x) = y_0(x)$.

Furthermore, the solution depends Lipschitz continuously on the coupling conditions. I.e. choosing different but similar coupling conditions does not affect the solution too much.

Compressor as special junctions:
Here, we are restricted to $n = 2$ and coupling conditions (3) with time dependent compressor control $P(t)$. Assume that the initial conditions are given by (y_0, P_0) and let $(\bar{y}, \bar{P})$ be a constant state fulfilling the coupling conditions. Define now

$$\mathcal{D}^\delta = \left\{(y, P) \in (\bar{y}, \bar{P}) + L^1\left(\mathbb{R}^+, \Omega \times \mathbb{R}^n\right) : TV((y, P)) < \delta\right\}$$

for some $\delta > 0$.
Then there exist δ, T and a semigroup

$$\mathcal{E} : [0, T - t_0] \times [0, T] \times \mathcal{D}_{t_0} \to \mathcal{D}^\delta$$

such that $\mathcal{E}(t, t_0, y_0, P)$ is the solution of the Cauchy problem at the junction at $t = t_0 + t$ with initial condition $y(t_0, x) = y_0(x)$, $P(t_0) = P_0$.
Furthermore, the solution depends Lipschitz continuous on the compressor control P. I.e.

$$\|\mathcal{E}(t, t_0, y_0, P) - \mathcal{E}(t, t_0, \tilde{y}_0, \tilde{P})\| \leq L \cdot \left(\|y_0 - \tilde{y}_0\| + \int_{t_0}^{t_0+t} \|P(\tau) - \tilde{P}(\tau)\| d\tau\right)$$

Remark:
The result proven in [2] is a lot more general and comprises a whole class of coupling conditions. Furthermore, both results also hold true for the flow of liquids in open canals as long as the flow can be described by the shallow water equations.

Optimal Control

Theorem:
Let $n \in \mathbb{N}$, $n \geq 2$ and define

$$\mathcal{P} = \left\{P \in \bar{P} + L^1\left([0,T]; \mathbb{R}^n\right) : (y_0, P) \in \mathcal{D}^\delta\right\}.$$

Assume that
$\mathcal{J}_0 : \left\{P : P \in \mathcal{P} \text{ and } (y_0, P) \in \mathcal{D}^\delta\right\} \to \mathbb{R}$ and $\mathcal{J}_1 : \mathcal{D}^\delta \to \mathbb{R}$ are both non–negative and lower semicontinuous with respect to the L^1-norm.
Then the cost functional

$$J(P) = J_0(P) + \int_0^T J_1(\mathcal{E}(t, t_0, y_0, P))dt$$

admits a unique minimum on $\mathcal{P}$.

Example: Application to gas flow:
Typical optimization problems in the control of gas pipelines have to penalize

- large energy consumption
- large deviations from the desired outlet pressure $\bar{p}$
- frequent changes in the applied compressor power

This can be achieved throught the following choices of $\mathcal{J}_0$ and $\mathcal{J}_1$:

$$\mathcal{J}_0(P) = TV(P) + \|P\|_{L^\infty}$$
$$\mathcal{J}_1(\mathcal{E}) = \int_{x_a}^{x_b} |p(\rho^{(2)}(t,x)) - \bar{p}|dx,$$

where $\rho^{(2)}$ is the solution of the Cauchy problem in the outgoing pipe.

Clearly $\mathcal{J}_0$ is lower semicontinuous and $\mathcal{J}_1$ is L^1-Lipschitz continuous. Therefore, the above Theorem applies and the existence of an optimal control in proven.

Remark:
Similar results can be obtained for the flow of liquids in open canals with underflow gates, valves or pumipng stations.

References

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Funding DFG SPP 1253, D/08/11076 and Seed Funds RWTH Aachen, Germany.