Modulational Instability of Periodic Waves

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Generalized KdV Equation

Consider the generalized Korteweg-de Vries equation:

$$u_t = u_{xxx} + f(u)_x.$$

Such equations arise naturally in applications:

- $f(u) = u^2$ corresponds to the KdV, which models 1-D shallow water waves.
- $f(u) = u^2 + \alpha u^3$ corresponds to the Gardner equation, which arises in modeling tide generated internal waves.
- $f(u) = \pm u^3$ arises in plasma physics and describes propagation of interfacial waves in a stratified medium.

Solutions stationary in a moving frame $x + ct$ with positive wave speed $c > 0$ are called traveling waves, and such profiles satisfy the traveling wave ODE

$$u_{xxx} + f(u)_x - cu_x = 0$$

We are interested in studying the stability of the periodic solutions of the traveling wave ODE to long-wavelength perturbations, i.e. to slow modulations of the underlying wave.

Integrability of Traveling Wave ODE

Traveling wave ODE reducible to quadrature:

$$\frac{u_x^2}{2} = E + au + \frac{c}{2}u^2 - F(u)$$

where $F' = f$, $(E, a) \in \mathbb{R}^2$. If the effective potential $V(x) = F(x) - ax - \frac{c}{2}x^2$ has a local minima, we are guaranteed the existence of periodic orbits. Gives parameter regime $\Omega \subset \mathbb{R}^3$ which parameterizes such solutions.
Periodic traveling waves of gKdV form a four parameter family:

$$u(x + x_0; a, E, c)$$

where x_0 represents the translation invariance. Let $T = T(a, E, c)$ denotes the corresponding period. Modding out this continuous symmetry, we are left with a three parameter family of such solutions.

Variational interpretation of a and c: Define Hamiltonian energy

$$\mathcal{H}(\phi) = \int_0^T \frac{1}{2}\phi_x^2 - F(\phi)\, dx.$$

Then traveling waves are critical points of $\mathcal{H}$ under the constraint that the mass and momentum functionals

$$M(\phi) = \int_0^T \phi dx, \quad P(\phi) = \int_0^T \phi^2 dx$$

are conserved, with a and c representing Lagrange multipliers associated with these functionals

Linearization & Periodic Evans Function

Fixing $(a, E, c) \in \Omega$ and linearizing traveling gKdV with respect to localized perturbations leads to spectral problem

$$\partial_x \mathcal{L}[u]v = \mu v \tag{1}$$

on $L^2(\mathbb{R})$, where $\mathcal{L}[u] = -\partial_x^2 - f'(u) + c$ is a periodic Hill operator. Floquet theory implies $\partial_x \mathcal{L}[u]$ has no discrete spectrum $\Rightarrow \sigma(\partial_x \mathcal{L}[u])$ is purely continuous!

Weyl sequence argument $\Rightarrow \sigma(\partial_x \mathcal{L}[u])$ consists of L^∞ eigenvalues. To classify such μ, write (1) as a first order system

$$\Phi' = H(x, \mu)\Phi, \quad \Phi(0, \mu) = \mathbf{I} \tag{2}$$

and let $\mathbf{M}(\mu) = \Phi(T, \mu)$ be the corresponding monodromy matrix. Then $\mu \in \sigma(\partial_x \mathcal{L}[u])$ iff $\mathbf{M}(\mu)$ has an eigenvalue on the unit circle, i.e. iff $\exists \lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that

$$D(\mu, \lambda) := \det(\mathbf{M}(\mu) - \lambda \mathbf{I})$$

vanishes. We call $D(\mu, \lambda)$ the periodic Evans function.

Local Analysis near $\mu = 0$

Noether's theorem implies infinitesimal variations in x_0, a, E, and generate solutions of the equation $\partial_x \mathcal{L}[u]v = 0$. Indeed,

$$\mathcal{L}[u]u_x = 0, \quad \mathcal{L}[u]u_a = -\frac{\delta \mathcal{M}}{\delta a} = -1, \quad \mathcal{L}[u]u_E = 0.$$

Follows that $0 \in \operatorname{spec}(\partial_x \mathcal{L}[u])$, and in particular $D(\mu, 1) = 0$. Moreover,

$$\mathcal{L}[u]u_c = -\frac{\delta \mathcal{P}}{\delta c} = -u$$

and hence variations in c gives information about the Jordan chain in the translation direction. Notice this reflects the fact that the constants a and c arise as Lagrange multipliers to enforce the mass and momentum constraints, respectively.

Goal: Study $\operatorname{spec}(\partial_x \mathcal{L}[u])$ near $\mu = 0$.

Can use u_x, u_a, u_E to construct monodromy operator $\mathbf{M}(\mu)$ at $\mu = 0$. Variation in c useful in following perturbation argument, as it gives the $\mathcal{O}(|\mu|)$ variation in the translation direction u_x.

In a neighborhood of $(\mu, \kappa) = (0, 0)$, have asymptotic expansion

$$D(\mu, e^{i\kappa}) = i\kappa^3 + \frac{i\kappa\mu^2}{2}\left(\{T, P\}_{E,c} + 2\{M, P\}_{E,a}\right) - \frac{\mu^3}{2}\{T, M, P\}_{a,E,c} + \mathcal{O}(|\mu|^4 + \kappa^4)$$

where the notation

$$\{F, G\}_{x,y} := \frac{\partial(F, G)}{\partial(x, y)}, \quad \{F, G, R\}_{x,y,z} := \frac{\partial(F, G, R)}{\partial(x, y, z)}.$$

is used to denote Jacobians of particular maps.
If $\mu = i\alpha\kappa + \mathcal{O}(\kappa^2)$, it follows that α must be a root of the polynomial

$$P(y) = 1 - \frac{y^2}{2}\left(\{T, P\}_{E,c} + 2\{M, P\}_{E,a}\right) - \frac{y^3}{2}\{T, M, P\}_{a,E,c}.$$

Modulational stability determined by discriminant of P: Define

$$\Delta_{MI} := \frac{1}{2}\left(\{T, P\}_{E,c} + 2\{M, P\}_{E,a}\right)^3 - \frac{27}{4}\{T, M, P\}_{a,E,c}^2.$$

- $\Delta_{MI} > 0 \Rightarrow$ Modulational Stability. Locally, $\operatorname{spec}(\partial_x \mathcal{L}[u])$ consists of a symmetric interval on $\mathbb{R}i$ about $\mu = 0$ of multiplicity three.
- $\Delta_{MI} < 0 \Rightarrow$ Modulational Instability. Locally, $\operatorname{spec}(\partial_x \mathcal{L}[u])$ consists of a symmetric interval on $\mathbb{R}i$ about $\mu = 0$ of multiplicity one, along with two branches of spectrum bifurcating from the origin along straight lines of non-zero slope.

Analysis near Homoclinic Wave

If we restrict to $f(u) = u^{p+1}$ for $p \geq 1$, the induced scaling in the wave speed parameter c implies that near a homoclinic wave one has

$$\Delta_{MI} \sim \frac{1}{2}\left(\left(\frac{4-p}{2pc}\right)\frac{\partial T}{\partial E}P(a, E, c)\right)^3.$$

Since $\frac{\partial T}{\partial E} > 0$ for dnoidal type solutions, solutions below seperatrix near homoclinic orbit satisfy

- $p < 4 \Rightarrow$ Modulationally Stable.
- $p > 4 \Rightarrow$ Modulationally Unstable.

Moreover, since $\frac{\partial T}{\partial E} < 0$ for cnoidal type solutions just above the separatrix it follows that such solutions near the homoclinic orbit are modulationally unstable when $p < 4$ and modulationally stable when $p > 4$: for example, see the focusing mKdV example in the next section.
Note: Such a modulational instability was not predicted by Gardner's long-wavelength theory. There, one was concerned with eigenvalues of the linearized operator associated with the limiting homoclinic wave. Instability was deduced by this operator having unstable eigenvalues. However, $\mu = 0$ is only a *marginally* stable eigenvalue and it was not clear from Gardner's theory if such an eigenvalue would contribute to spectral instability.

Computation of Δ_{MI}: KdV and mKdV

In the case of KdV ($f(u) = u^2$) or mKdV ($f(u) = u^3$ for focusing and $f(u) = -u^3$ for defocusing), the period and conserved quantities M and P can be represented as Abelian integrals of first, second, and third kind on a Riemann surface:

$$T = \sqrt{2}\oint \frac{du}{\sqrt{R(u)}}, \quad M = \sqrt{2}\oint \frac{u\, du}{\sqrt{R(u)}}, \quad P = \sqrt{2}\oint \frac{u^2\, du}{\sqrt{R(u)}}$$

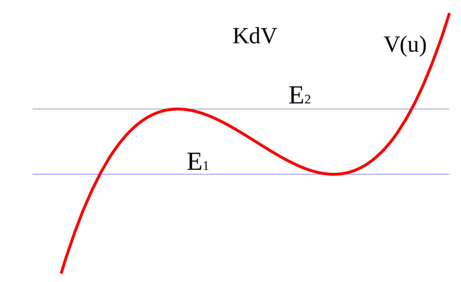
where the integral is taken around the (compact) classically allowed region and

- $R(u) = E + au + \frac{c}{2}u^2 - u^3/3$ for KdV,
- $R(u) = E + au + \frac{c}{2}u^2 - u^4/4$ for focusing mKdV,
- $R(u) = E + au + \frac{c}{2}u^3 + u^4/4$ for defocusing mKdV.

For KdV, standard Elliptic integral calculations show

$$\Delta_{MI} = \frac{\Omega_1^2}{16 \cdot 12^3 \operatorname{disc}(R)^3}$$

where $\operatorname{disc}(R)$ is the discriminant of the cubic polynomial $R(u)$, where Ω_1 is a nonlinear combination of a, E, c, T, and M. The plot of the effective potential V for a fixed a and c in this case is shown below.



The classically allowed region consists of energy levels $E \in [E_1, E_2]$, and for such E the polynomial R has three real roots. Thus, all periodic traveling wave solutions of KdV are modulationally stable!!

For mKdV, the story is more interesting. In the defocusing case one has

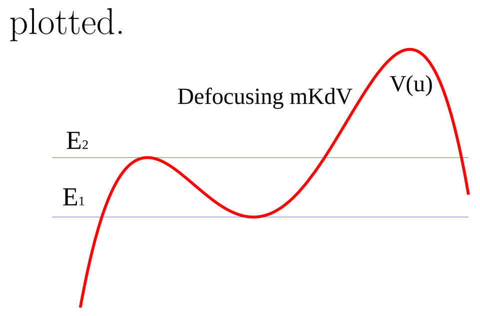
$$\Delta_{MI} = \frac{\Omega_2^2}{16^4 \operatorname{disc}(R)^3},$$

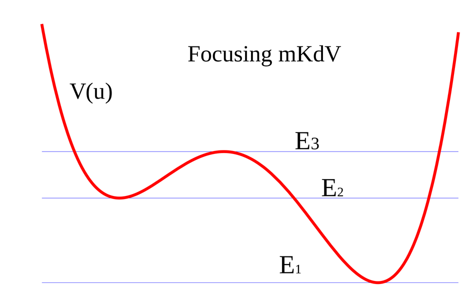
where Ω_2 is a nonlinear combination of a, E, c, T, and P. A graph of the effective potential is shown below on the left. The classically allowed region is now the small well between energy levels E_1 and E_2, and for all such energies the equation $R(u) = 0$ has four real solutions. Hence such periodic solutions are always modulationally stable.

In the focusing case one has

$$\Delta_{MI} = \frac{\Omega_3^2}{16^4 \operatorname{disc}(R)^3},$$

where Ω_3 is a nonlinear combination of a, E, c, T, and P but now $\operatorname{disc}(R)$ depends on the particular $(a, E, c) \in \Omega$ chosen. Indeed, consider the figure below on the right where an effective potential V is plotted.





If one considers periodic solutions with energy levels $E \in (E_2, E_3)$, then the discriminant of the polynomial $R = E - V(x)$ must be positive. Hence, for such solutions are always modulational stable.
However, if one considers periodic solutions with energy levels $E \in (E_1, E_2)$ or $E > E_3$, then the discriminant of R is negative, and hence such solutions are always modulationally unstable.

Discussion

- Integrability of traveling wave ODE and perturbation theory yields geometric criterion for modulational instability of periodic waves via rigorous Whitham theory calculations.
- Geometric criterion can be (easily) computed numerically in case of polynomial nonlinearity using Abelian integral representation of moments of underlying wave.
- Methods extend to provide information about spectral and nonlinear stability to periodic perturbations, as well as transverse instabilities in higher dimensional models.