L^2 Stability Estimates for Shock Solutions of Scalar Conservation Laws Using the Relative Entropy Method

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Background

For scalar nonviscous conservation laws with general flux, it is well-known from the theory of Kružkov that the solution operator for the initial value problem

$$\begin{cases} \partial_t U + \partial_x A(U) = 0; \\ U(x,0) = U^0(x), \end{cases} \tag{1}$$

forms an L^1-contraction semigroup. As a result, the measure in $L^1(\mathbb{R})$ of the difference of any pair of entropy solutions is non-increasing over time. In particular, if $\phi(x-\sigma t)$ is a shock wave and $U^0 - \phi \in L^1(\mathbb{R})$, then

$$\| U(\cdot,t) - \phi(\cdot - \sigma t) \|_{L^1(\mathbb{R})} \leq \| U^0 - \phi \|_{L^1(\mathbb{R})} \tag{2}$$

for all $t \geq 0$.

On the other hand, for conservation laws with a convex extension, it is well-known that the relative entropy method developed by Dafermos and DiPerna provides L^2 stability estimates for solutions away from shocks. One gets estimates of the form

$$\| U(\cdot,t) - \overline{U}(\cdot,t) \|_{L^2([-R,R])} \leq a e^{bt} \| U^0 - \overline{U}^0 \|_{L^2([-R-st,R+st])}, \tag{3}$$

where U and $\overline{U}$ are weak and strong solutions, respectively, and where a, b, and s are constants depending on the initial data U_0 and $\overline{U}_0$. However, simple examples show that this kind of result cannot hold when $\overline{U}$ is a shock wave or more generally when $\overline{U}$ is only a weak solution of the conservation law. Nonetheless, we can get similar estimates up to a suitable translation of the shock.

Relative entropy

We say that $U \in L^\infty(\mathbb{R} \times (0,\infty))$ is an entropy solution of (1) if U satisfies

$$\partial_t \eta(U) + \partial_x G(U) \leq 0,$$

in the sense of measures, for all entropy/entropy-flux pairs $(\eta, G) \in [C^\infty(\mathbb{R})]^2$ verifying $\eta'' \geq 0$ and $G' = \eta' A'$. If $\overline{U}$ solves (1) in the classical sense, and we consider, associated to each convex η, the relative entropy function

$$\eta(U \mid \overline{U}) = \eta(U) - \eta(\overline{U}) - \eta'(\overline{U}) \cdot (U - \overline{U}),$$

then an entropy solution U verifies additionally the inequality

$$\begin{aligned} \partial_t \left[\eta(U \mid \overline{U})\right] + \partial_x \left[G(U) - G(\overline{U}) - \eta'(\overline{U}) \cdot (A(U) - A(\overline{U}))\right] \\ \leq -\partial_t \left[\eta'(\overline{U})\right] \cdot (U - \overline{U}) - \partial_x \left[\eta'(\overline{U})\right] \cdot (A(U) - A(\overline{U})), \end{aligned} \tag{4}$$

for the same entropy/entropy-flux pairs (η, G). In particular, when $\overline{U}(x,t) = C$, for constant states $C \in \mathbb{R}$, (4) reduces to

$$\partial_t \eta(U \mid C) + \partial_x F(U,C) \leq 0, \tag{5}$$

where

$$\begin{aligned} F(U,C) &= G(U) - G(C) - \eta'(C) \cdot (A(U) - A(C)) \\ &= \int_C^U (\eta'(w) - \eta'(C)) A'(w)\, dw, \end{aligned} \tag{6}$$

represents the flux of relative entropy.

The idea of the relative entropy method is to use (4) to estimate the quantity $\int \eta(U \mid \overline{U})\, dx$ in time. When $\eta(U) = U^2$, this corresponds directly to estimates in the L^2 metric, as given, for example, in (3).

The main result

Consider the initial value problem (1) for a scalar conservation law in one space dimension. We prove the the following global L^2 stability estimate for shocks.

Theorem 1. *Let $U^0 \in L^\infty(\mathbb{R})$ and assume $U^0 - \phi \in L^2(\mathbb{R})$ where*

$$\phi(x) = \begin{cases} C_L, & \text{if } x < 0; \\ C_R, & \text{if } x > 0, \end{cases} \tag{7}$$

with $C_L > C_R$. Further, assume U is the unique entropy solution of (1), for a smooth flux function $A : \mathbb{R} \to \mathbb{R}$ verifying $A'' > 0$. Then there exists a Lipschitz continuous function $\overline{x} : [0,\infty) \to \mathbb{R}$ and a constant $\lambda(\| U^0 \|_{L^\infty}; \phi; A) > 0$ such that

$$\| U(\cdot,t) - \phi(\cdot - \sigma t - \overline{x}(t)) \|_{L^2(\mathbb{R})} \leq \| U^0 - \phi \|_{L^2(\mathbb{R})} \tag{8}$$

and

$$|\overline{x}(t)| \leq \lambda \| U^0 - \phi \|_{L^2(\mathbb{R})} \sqrt{t} \tag{9}$$

for all $t \geq 0$, where σ is given by the relation $\sigma(C_L - C_R) = A(C_L) - A(C_R)$.

General strategy

The idea is to construct curves $x_L : [0,\infty) \to \mathbb{R}$ and $x_R : [0,\infty) \to \mathbb{R}$, initialized with the data $x_L(0) = x_R(0) = 0$, for which the the total relative entropy

$$\mathcal{E}(t) = \int_{-\infty}^{x_L(t)} \eta(U(x,t) \mid C_L)\, dx + \int_{x_R(t)}^{\infty} \eta(U(x,t) \mid C_R)\, dx = \mathcal{E}_L(t) + \mathcal{E}_R(t) \tag{10}$$

is bounded above by $\mathcal{E}(0)$ for all $t \geq 0$. Due to the compressive nature of the solution $U(x,t)$, the construction will produce automatically the constraint $x_R(t) \leq x_L(t)$ for all $t \geq 0$. Therefore, (8) will follow for any function $\overline{x}(t)$ satisfying $x_R(t) \leq \overline{x}(t) + \sigma t \leq x_L(t)$. In particular, this includes the curves $\overline{x}(t) = x_L(t) - \sigma t$ and $\overline{x}(t) = x_R(t) - \sigma t$.

Using (5), we obtain formally

$$\begin{cases} \dot{\mathcal{E}_L}(t) = \dot{x_L}(t) \eta(U(x_L(t),t) \mid C_L) - F(U(x_L(t),t), C_L); \\ \dot{\mathcal{E}_R}(t) = F(U(x_R(t),t), C_R) - \dot{x_R}(t) \eta(U(x_R(t),t) \mid C_R). \end{cases} \tag{11}$$

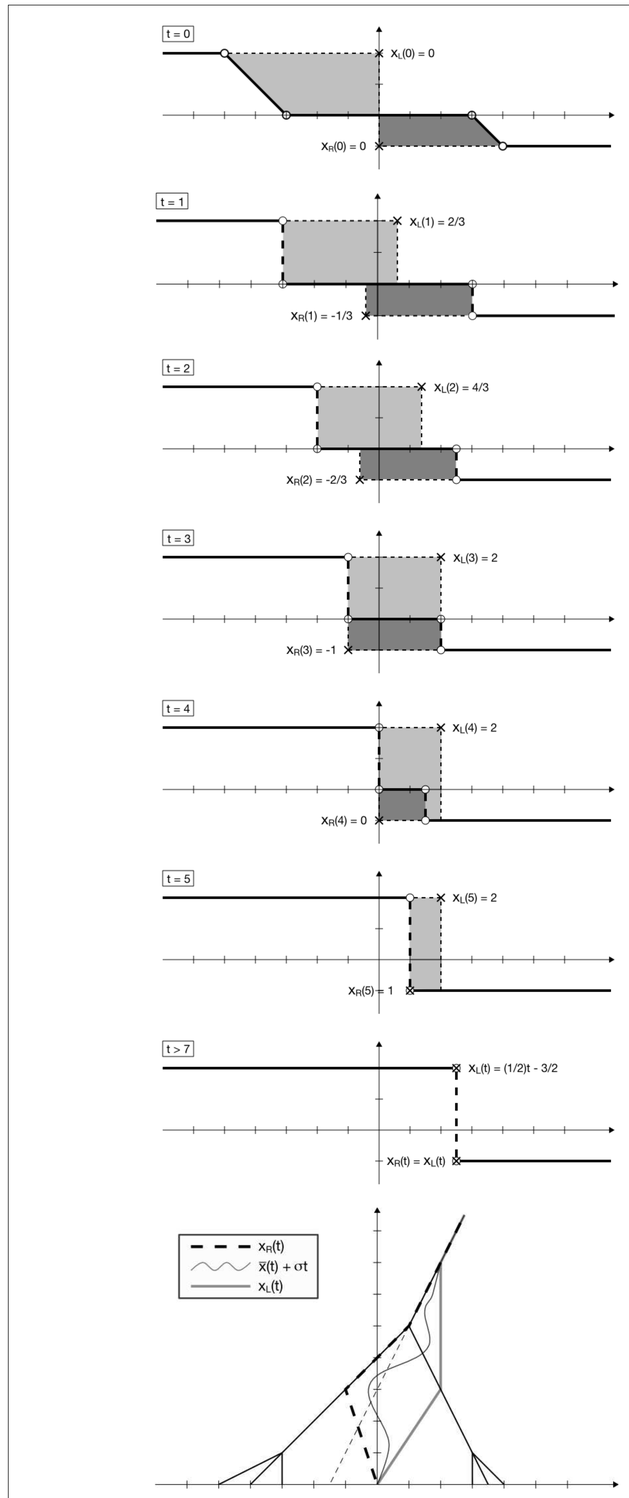
Therefore, a suitable choice of the functions x_L and x_R corresponds to solutions of

$$\begin{cases} \dot{x_L}(t) = f(U(x_L(t),t), C_L) \doteq \frac{F(U(x_L(t),t), C_L)}{\eta(U(x_L(t),t) \mid C_L)}; \\ \dot{x_R}(t) = f(U(x_R(t),t), C_R) \doteq \frac{F(U(x_R(t),t), C_R)}{\eta(U(x_R(t),t) \mid C_R)}, \end{cases} \tag{12}$$

in the sense of Filippov. The key observation is that f is increasing in its second argument so that if x_L and x_R intersect at $t = t_0$, we have

$$\dot{x_R}(t_0) = f(U(x_R(t_0),t_0), C_R) \leq f(U(x_L(t_0),t_0), C_L) = \dot{x_L}(t_0),$$

which formally justifies the claim that $x_R(t) \leq x_L(t)$ for all $t \geq 0$.

A simple example

Consider the inviscid Burgers' equation ($A(u) = \frac{u^2}{2}$), for which $f(U,C) = \frac{2}{3}U + \frac{1}{3}C$. The initial data U^0, shown in the first figure, is an L^2 perturbation of the shock with endstates $C_L = 2$ and $C_R = -1$. The evolution of $x_L(t)$ and $x_R(t)$ is shown in the series of graphs below. Note that until the appearance of shocks at $t = 1$, the total relative entropy is preserved; that is, $\mathcal{E}(t) = \mathcal{E}(0)$. Note, also that $\overline{x}(t) \to x_0 = -\frac{3}{2}$. the shift considered in the large-time stability analysis of Il'in and Oleinik.