On the motion of several rigid bodies in an incompressible non-Newtonian and heat-conducting fluid

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1 Introduction

Over the past years, there has been a great impulse in studying the motion of a rigid body. We would like to quote the work of Brenner [2] concerning the steady motion of one or more bodies in a linear viscous liquid in the Stokes approximation as well as Weinberger [18], [19], Serre [16] regarding the fall of a body in an incompressible Navier-Stokes fluid under the action of gravity and Borchers [1] for the existence of weak solutions. We would like to distinguish that there are two different problems. First one which is modelling the motion of fluid around a moving body with prescribed velocity see work of Galdi, Hishida, Farwig, [6, 7, 8, 9, 12]. The another problem is the motion of rigid bodies in a viscous fluids, which satisfies the second Newton's law and the conservation of momentum of momentum. In our paper we will deal with the second problem. We consider the motion of one or several rigid bodies in a viscous fluid occupying a bounded domain $\Omega \subset R^3$ represents an interesting theoretical problem featuring, among others, possible contacts of two or more solid objects. We will consider the problem of rigid bodies in viscous non-Newtonian fluids where both are heat conducting materials. Using penalization method developed by Conca, San Martin, Tucsnak and

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Starovoitov [15, 3], they have shown the existence of weak solution up to the first contact.

2 Formulation of the problem

2.1 Bodies and motions

A *rigid body* can be identified with a connected compact subset $\bar{\mathbf{S}}$ of the Euclidean space R^3. The motion of bodies can be represented as a mapping $\eta : (0,T) \times R^3 \to R^3$, where $\eta(t,\cdot) : R^3 \to R^3$ is an isometry for any *time* $t \in (0,T)$.

Remark:

We adopt the Eulerian (spatial) description of motion, where the coordinate system is attached to a fixed region of the physical space currently occupied by the fluid.

The *place* $\mathbf{x}$ and the *time* $t \in (0,T)$ play the role of independent variables. The mappings $\eta(t,\cdot)$ are isometries,

$$\eta(t,\mathbf{x}) = \mathbf{X}_g(t) + \mathcal{O}(t)(\mathbf{x} - \mathbf{X}_g(0)),$$

where $\mathbf{X}_g$ - the position of the center of mass at a time t, and $\mathcal{O}(t)$ is a matrix satisfying $\mathcal{O}^T\mathcal{O} = \mathcal{I}$. The translation and angular velocities satisfy in the Eulerian coordinate the following equations

$$\frac{d}{dt}\mathbf{X}_g = \mathbf{U}_g \text{ - the translation velocity,} \tag{2.1}$$

and

$$\frac{d}{dt}\mathcal{O}(t)\mathcal{O}^T(t) = \mathcal{Q}(t) \text{ - the angular velocity.} \tag{2.2}$$

Accordingly, the solid velocity in the Eulerian coordinate system can be written in the form

$$\mathbf{u}^S(t,\mathbf{x}) = \frac{\partial \eta}{\partial t}(t, \eta^{-1}(t,\mathbf{x})) = \mathbf{U}_g(t) + \mathcal{Q}(t)(\mathbf{x} - \mathbf{X}_g(t)).$$

Under the assumptions of continuity of the stress, the balance of linear and angular momentum for the body S_i

$$m\frac{d}{dt}\mathbf{U}_g(t) = \frac{d}{dt}\int_{\mathbf{S}(t)} \rho^S \mathbf{u}^S \, d\mathbf{x} = \int_{\mathbf{S}(t)} \rho^S \mathbf{g}^S \, \mathrm{d}x + \int_{\partial\mathbf{S}(t)} \mathcal{T}\mathbf{n} \, d\sigma, \tag{2.3}$$

$$\frac{d}{dt}(\mathcal{J}\omega) = \frac{d}{dt}\int_{\mathbf{S}(t)} \rho^S (\mathbf{x} - \mathbf{X}_g) \times \mathbf{u}^S \, \mathrm{d}x = \tag{2.4}$$

$$\int_{\partial\mathbf{S}(t)} (\mathbf{x} - \mathbf{X}_g) \times \mathcal{T}\mathbf{n}\, \mathrm{d}\sigma + \int_{\mathbf{S}(t)} \rho^S (\mathbf{x} - \mathbf{X}_g) \times \mathbf{g}^S\, \mathrm{d}x,$$

with $\mathcal{J}$ - the *inertial tensor*

$$\mathcal{J}\mathbf{a} \cdot \mathbf{b} = \int_{\mathbf{S}(t)} \rho^S (\mathbf{a} \times (\mathbf{x} - \mathbf{X}_g)) \cdot (\mathbf{b} \times (\mathbf{x} - \mathbf{X}_g))\, \mathrm{d}x$$

where m is the total mass of the body, the angular velocity $\mathcal{Q}$ is skew-symmetric is such that there exists a vector ω such that

$$\mathcal{Q}(t)(\mathbf{x} - \mathbf{X}_g) = \omega(t) \times (\mathbf{x} - \mathbf{X}_g).$$

The position of body at time $t \in [0, T]$ would be given as

$$\bar{\mathbf{S}}(t) = \eta(t, \bar{\mathbf{S}}),$$

where $\eta(t, x)$, $t \in [0, T]$ is the characteristic curve determined uniquely through the initial value problem

$$\frac{\partial}{\partial t}\eta(t, x) = u(t, \eta(t, x)), \eta(0, x) = x.$$

2.2 The fluid motion

The fluid is completely determined by its density ρ^f, the velocity $\mathbf{u}^f$ and the temperature θ^f. The standard mass and momentum balance and energy equations read

$$\partial_t \rho^f + \mathrm{div}(\rho^f \mathbf{u}^f) = 0, \tag{2.5}$$

$$\mathrm{div}\mathbf{u}^\mathbf{f} = 0$$

$$\partial_t(\rho^f \mathbf{u}^f) + \mathrm{div}(\rho^f \mathbf{u}^f \otimes \mathbf{u}^f) + \nabla p = \mathrm{div}\mathcal{S} + \rho^f \mathbf{h}^f + \rho^f \mathbf{G}\theta^f, \tag{2.6}$$

$$\rho^f (\partial_t(\theta^f) + \mathbf{u}^f \cdot \nabla\theta^f) + \mathrm{div}\ (q^f) = 0, \tag{2.7}$$

where
p is the pressure,

$\mathbf{h}^f$ is the specific body force,

$\mathbf{G}$ is the buoyancy force and in the Boussinesq approximation is described by

$\mathbf{G} = \gamma(0,0,g)^\top$,
γ denotes the coefficient of thermal dilatation
g is the constant of gravity.
$\mathcal{S}$ is the viscous stress tensor,
θ^f is the absolute temperature,
q^f is the heat flux satisfying the modified Fourier law
$q = \kappa \nabla\theta\theta^\beta$,
κ the coefficient of heat conductivity
$\beta \in \mathbf{R}$.

We will consider the following boundary conditions

$$\begin{aligned} u^f &= 0 \quad \text{on } \partial\Omega, \\ \theta^f &= 0 \quad \text{on } \Gamma_D, \\ \kappa\tfrac{\partial q^f}{\partial n} &= h \quad \text{on } \Gamma_N. \end{aligned} \tag{2.8}$$

2.3 Viscous stress constitutive relation

$$\mathcal{S} = \mathcal{S}[\ \mathbf{D}[\mathbf{u}]\ ],\ \mathcal{S} : R^{3\times3}_{\text{sym}} \to R^{3\times3}_{\text{sym}} \text{ continuous}, \tag{2.9}$$

$$\Big(\mathcal{S}[\mathbf{M}] - \mathcal{S}[\mathbf{N}]\Big) : \Big(\mathbf{M} - \mathbf{N}\Big) > 0 \text{ for all } \mathbf{M} \neq \mathbf{N}, \tag{2.10}$$

$$c_1|\mathbf{M}|^p \leq \mathcal{S}[\mathbf{M}] : \mathbf{M} \leq c_2|\mathbf{M}|^p \text{ for } p \geq 4. \tag{2.11}$$

$$Q := I \times \Omega,$$

$$Q^i = \{(t,x)|t \in I, x \in \bar{S}^i(t)\}$$

$$Q^s := \bigcup_{i=1}^{N} Q^i, Q^f := Q \setminus Q^s$$

We define the following quantity:

$$\rho(t,\mathbf{x}) = \begin{matrix} \rho^f(t,\mathbf{x}) \text{ on} Q^f \\ \rho^{S_i}(t,\mathbf{x} \text{ on } Q^i \\ 0 \text{ on } R^3 \setminus \Omega \end{matrix}$$

$$u(t,\mathbf{x}) \begin{matrix} = u^f(t,\mathbf{x}) \text{ on } Q^f \\ = u^{S_i}(t,\mathbf{x}) \text{ on } Q^i \\ = 0 \text{ on } R^3 \setminus \Omega \end{matrix}$$

$$\theta(t,\mathbf{x}) \begin{matrix} = \theta^f \text{ on } Q^f \\ = \theta^i \text{ on } Q^i \end{matrix}$$

Assumption:

$$\kappa^f = \kappa^{S^i}$$

3 Preliminaries, weak formulation

The initial position of the rigid bodies is determined through a family of domains

$$S_i \subset R^3, \ i = 1, \dots, n,$$

each of them being diffeomorphic to the unit ball in R^3. In addition, in order to facilitate the analysis, the boundaries of all rigid bodies are assumed to be regular, more specifically, there exists $\delta_0 > 0$ such that for any $x \in \partial S_i$, there are two closed balls $B^{\rm int}$, $B^{\rm ext}$ of the radius δ_0 such that

$$x \in B^{\rm int} \cap B^{\rm ext}, \ B^{\rm int} \subset \overline{S}_i, \ B^{\rm ext} \subset R^3 \setminus S_i \tag{3.1}$$

Similarly, the underlying physical space $\Omega \subset R^3$, occupied by the fluid containing the rigid bodies, is supposed to be a domain such that for any $x \in \partial\Omega$, there are two closed balls $B^{\rm int}$, $B^{\rm ext}$ of the radius δ_0 such that

$$x \in B^{\rm int} \cap B^{\rm ext}, \ B^{\rm int} \subset \overline{\Omega}, \ B^{\rm ext} \subset R^3 \setminus \Omega. \tag{3.2}$$

The motion η_i associated to the body S_i is a mapping

$$\eta_i = \eta_i(t,x), \ t \in [0,T), \ x \in R^3, \ \eta_i(t,\cdot) : R^3 \to R^3 \text{ is an isomorphismus,}$$

$$\eta_i(0,x) = x \text{ for all } x \in R^3, \quad i = 1, \dots, n.$$

Accordingly, the position of the body S_i at a time t is given through formula

$$S_i(t) = \eta_i(t, S_i), \ i = 1, \dots, n.$$

The present work relies on the widely used concept of *weak solution* introduced by Judakov [17] based on the Eulerian reference system and a class of test functions depending on the position of the rigid bodies (see Desjardins and Esteban [4, 5], Galdi [11],[12], Hoffmann and Starovoitov [14], San Martin et al. [15], Serre [16], among others). Specifically, the mass density $\varrho = \varrho(t,x)$ and the velocity field $\mathbf{u} = \mathbf{u}(t,x)$ at a time $t \in (0,T)$ and the spatial position $x \in \Omega$ satisfy the integral identities

$$\int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big) \ {\rm d}x \ {\rm d}t = - \int_\Omega \varrho_0 \varphi \ {\rm d}x \tag{3.3}$$

for any test function $\varphi \in C^1([0,T) \times \overline{\Omega})$, and

$$\int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \mathbb{D}[\varphi] - \mathbb{S} : \mathbb{D}[\varphi] \Big) \ {\rm d}x \ {\rm d}t \tag{3.4}$$

$$= -\int_0^T \int_\Omega \varrho \nabla_x F \cdot \varphi \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi \ \mathrm{d}x + \int_0^T \int_\Omega \varrho G\theta \cdot \varphi \ \mathrm{d}t \ \mathrm{d}x$$

for any test function $\varphi \in C^1([0,T) \times \overline{\Omega})$

$$\varphi(t,\cdot) \in [\mathcal{RM}](t), \tag{3.5}$$

where

$$[\mathcal{RM}](t) = \{\phi \in C^1(\overline{\Omega}) \mid \mathrm{div}_x \phi = 0 \text{ in } \Omega,\ \phi = 0 \text{ on a neighborhood of } \partial\Omega, \tag{3.6}$$

$h = \nabla F$ and

$$\mathbb{D}[\phi] = 0 \text{ on a neighborhood of } \cup_{i=1}^n \overline{S}_i(t)\}.$$

and

$$\int_0^T \int_\Omega \Big(\rho\theta \cdot \partial_t \psi + \rho \mathbf{u} \theta \nabla \psi + q \nabla \psi\Big) dx dt \quad = \quad \int_0^T \int_{\Gamma_N} h\psi \ dS \ dt \tag{3.7}$$

for any test function $\psi \in C^1([0,T) \times \bar{\Omega})$,

$$\psi(t,.) \in [\mathcal{C}](t) \tag{3.8}$$

$$[\mathcal{C}](t) = \{\psi \in C^1(\bar{\Omega}) \mid \text{in } \Omega,\ \psi = 0 \text{ on a neighborhood of } \ \Gamma_D\}, \tag{3.9}$$

Here, the symbol $\mathbb{S}$ stands for the viscous stress tensor determined through (2.9 - 2.11), $\nabla_x F$ is a given potential driving force, and ϱ_0, $\mathbf{u}_0$, θ_0 denote the initial distribution of the density, the velocity and the temperature, respectively.q is the heat flux obeying Fourier's law

$$q = -\frac{\kappa}{\beta+1} \nabla_x \theta^{\beta+1},$$

with the heat conductivity coefficient κ.

Finally, we require the velocity field $\mathbf{u}$ to be compatible with the motion of the rigid bodies. This can be formulated as follows. As the mappings $\eta_i(t,\cdot)$ are isometries on R^3, they can be written in the form

$$\eta_i(t,x) = x_i(t) + \mathbb{O}_i(t)x.$$

Accordingly, we shall say that the velocity field $\mathbf{u}$ is compatible with the family of motions $\{\eta_1, \ldots, \eta_n\}$ if

$$\mathbf{u}(t,x) = \mathbf{u}^{S_i}(t,x) = \mathbf{U}_i(t) + \mathbb{Q}_i(t)(x - x_i(t)) \text{ for a.a. } x \in \overline{S}_i(t),\ i = 1, \ldots, n \tag{3.10}$$

for a.a. $t \in [0,T)$, where

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i = \mathbf{U}_i,\ \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{O}_i\Big)\mathbb{O}_i^T = \mathbb{Q}_i \text{ a.a. on } (0,T). \tag{3.11}$$

Also we consider the continuity of the heat flux

$$q^f \cdot n[t] = q^s \cdot n[t] \text{ on } \partial S_i(t) \tag{3.12}$$

and also we consider that

$$\kappa^f = \kappa^{S_i}. \tag{3.13}$$

Note that an alternative albeit completely equivalent weak formulation of the problem may be found in Gunzburger et al. [13].

4 Main result

Theorem 4.1 *Let the initial position of the rigid bodies be given through a family of open sets*

$$S_i \subset \Omega \subset R^3,\ S_i \textit{ diffeomorphic to the unit ball for } i = 1,\dots,n,$$

where both ∂S_i, $i = 1,\dots,n$, and $\partial\Omega$ belong to the regularity class specified in (3.1), (3.2). In addition, suppose that

$$\mathrm{dist}[\overline{S}_i, \overline{S}_j] > 0 \textit{ for } i \neq j,\ \mathrm{dist}[\overline{S}_i, R^3 \setminus \Omega] > 0 \textit{ for any } i = 1,\dots,n.$$

*Furthermore, let the viscous stress tensor $\mathbb{S}$ satisfy hypotheses (**??** - **??**), with $p \geq 4$, and let $F \in W^{1,\infty}(\Omega)$. Finally, let the initial distribution of the density be given as*

$$\varrho_0 = \begin{cases} \varrho_f = \mathrm{const} > 0 \textit{ in } \Omega \setminus \cup_{i=1}^n \overline{S}_i, \\ \\ \varrho_{S_i} \textit{ on } S_i,\ \textit{where } \varrho_{S_i} \in L^\infty(\Omega),\ \mathrm{ess\,inf}_{S_i}\,\varrho_{S_i} > 0,\ i = 1,\dots,n, \end{cases}$$

while

$$\mathbf{u}_0 \in L^2(\Omega; R^3),\ \mathrm{div}_x \mathbf{u}_0 = 0 \textit{ in } \mathcal{D}'(\Omega),\ \mathbb{D}[\mathbf{u}_0] = 0 \textit{ in } \mathcal{D}'(S_i; R^{3\times 3}) \textit{ for } i = 1,\dots,n,$$

$$\theta_0 \in L^1(\Omega), \tag{4.1}$$

$$0 < \theta_* \leq \theta_0 \textit{ for a.a. } x \in \Omega$$

and $\beta > -min\{\frac{2}{3}, \frac{3p-5}{3(p-1)}\}$*. Then there exist a density function* ϱ*,*

$$\varrho \in C([0,T]; L^1(\Omega)),\ 0 < \operatorname{ess}\inf_{\Omega} \varrho(t,\cdot) \le \operatorname{ess}\sup_{\Omega} \varrho(t,\cdot) < \infty \text{ for all } t \in [0,T],$$

a family of isometries $\{\eta_i(t,\cdot)\}_{i=1}^n$*,* $\eta_i(0,\cdot) = \mathrm{I}$*, and a velocity field* $\mathbf{u}$*,*

$$\mathbf{u} \in C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^3)) \cap L^p(0,T; W_0^{1,p}(\Omega; R^3)),$$

compatible with $\{\eta_i\}_{i=1}^n$ *in the sense specified in (3.10), (3.11),*$\theta \in L^\infty(0,T; L^1(\Omega))$ *and* $\theta^{\frac{\beta-\lambda+1}{2}} L^2(0,T; W^{1,2})$ *for all* $\lambda \in (0,1)$ *such that* ϱ*,* $\mathbf{u}$ *satisfy the integral identity (3.3) for any test function* $\varphi \in C^1([0,T) \times R^3)$*, and the integral identity (3.4) for any* φ *satisfying (3.5), (3.6) and energy equation (the integral identity) holds for any admissible test function* $\vartheta \in \mathcal{C}(t)$*.*

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