⋆ Long-time stability of noncharacteristic boundary layers in gas dynamics ⋆

Toan Nguyen⋆ and Kevin Zumbrun
Indiana University
http://mypage.iu.edu/ ∼nguyentt

Boundary layers

Consider a laminar boundary layer

$$\begin{cases} (\rho,u,e)(x,t) = (\bar\rho,\bar u,\bar e)(x_1/\epsilon) \\ \lim\limits_{x_1\to+\infty}(\bar\rho,\bar u,\bar e)(x_1) = (\rho_+,u_+,e_+), \end{cases}$$

the planar stationary solution of the compressible Navier–Stokes equations

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0 \\ (\rho u)_t + \mathrm{div}(\rho u^t u) + \nabla p = \epsilon\mu\Delta u + \epsilon(\mu+\eta)\nabla \mathrm{div} u \\ (\rho E)_t + \mathrm{div}\big((\rho E + p)u\big) = \epsilon\kappa\Delta T + \epsilon\mu \mathrm{div}\big((u\cdot\nabla)u\big) \\ \qquad\qquad + \epsilon(\mu+\eta)\nabla(u\cdot \mathrm{div} u), \end{cases} \tag{1}$$

$x\in\mathbb{R}^d$, on a half-space $x_1>0$, where ρ denotes density, $u\in\mathbb{R}^d$ velocity, e specific internal energy, $E = e + \frac{|u|^2}{2}$ specific total energy, $p = p(\rho,e)$ pressure, $T = T(\rho,e)$ temperature; $\epsilon = 1/\mathrm{Re}$.

Boundary conditions

Equations (1) are considered with no-slip *suction-type* boundary conditions on the velocity,

$$u_j(0,x_2,\dots,x_d) = 0,\ j\neq 1 \quad \text{and} \quad u_1(0,x_2,\dots,x_d) = V(x) < 0,$$

and prescribed temperature, $T(0,x_2,\dots,x_d) = T_{wall}(\tilde x)$ (density on the wall is determined by the initial data); the outflow problem.
Or alternatively, with *blowing-type* boundary conditions

$$u_j(0,x_2,\dots,x_d) = 0,\ j\neq 1 \quad \text{and} \quad u_1(0,x_2,\dots,x_d) = V(x) > 0,$$

and prescribed temperature and pressure

$$T(0,x_2,\dots,x_d) = T_{wall}(\tilde x), \quad p(0,x_2,\dots,x_d) = p_{wall}(\tilde x)$$

(equivalently, prescribed temperature and density); the inflow problem.
This corresponds to the situation of an airfoil with microscopic holes through which gas is pumped from the surrounding flow, the microscopic suction imposing a fixed normal velocity while the macroscopic surface imposes standard temperature conditions as in flow past a (nonporous) plate. This configuration was suggested by Prandtl and tested experimentally by G.I. Taylor as a means to reduce drag by stabilizing laminar flow. It was implemented in the NASA F-16XL experimental aircraft program in the 1990's with reported 25% reduction in drag at supersonic speeds.

Possibility of boundary layers

Boundary layers often appear in small viscosity, i.e., $\epsilon\to 0$. Technically, the possibility is due to loss of number of necessary imposed boundary conditions when viscous and inviscid equations are considered. For instance, number of BCs of (1) is either $d+2$ or $d+1$ depending on values of $u_1(0,\tilde x)$ described above, whereas number of BCs for the Euler equations (i.e., $\epsilon = 0$) is equal to number of positive characteristic speeds.

The problem

- We study *long-time linearized and nonlinear stability* of such a boundary layer, under small initial and boundary perturbations. Boundary layers are assumed to be noncharacteristic: the signals to be transmitted into or out of but not along the boundary (i.e., $u_1(0,\tilde x)\neq 0$).
- The problem has been investigated for small amplitude isentropic layers by Matsumura-Nishihara (2001) and Kawashima-Nishibata-Zhu (2003) for one-d inflow/outflow case, and Kagei-Kawashima (2006) for multi-d case via energy estimate approach.

General form

The system will be written in a general form of hyperbolic–parabolic conservation laws

$$\tilde U_t + \sum_j F^j(\tilde U)_{x_j} = \sum_{jk}(B^{jk}(\tilde U)\tilde U_{x_k})_{x_j}, \quad x\in\mathbb{R}^d_+ = \{x_1>0\}, \quad t>0, \tag{2}$$

$\tilde U, F^j\in\mathbb{R}^n$, $B^{jk}\in\mathbb{R}^{n\times n}$, with initial data $\tilde U(x,0) = \tilde U_0(x)$.

Symmetrizability - extended Kawashima class

We assume that equations (2) can be written alternatively, after a triangular change of coordinates

$$\tilde W := \tilde W(\tilde U) = \begin{pmatrix} \tilde w^I(\tilde u) \\ \tilde w^{II}(\tilde u,\tilde v) \end{pmatrix}, \tag{3}$$

in *the quasilinear, partially symmetric hyperbolic-parabolic form*

$$\tilde A^0\tilde W_t + \sum_j \tilde A^j\tilde W_{x_j} = \sum_{jk}(\tilde B^{jk}\tilde W_{x_k})_{x_j} + \tilde G. \tag{4}$$

There are great advantages of using the symmetric form in obtaining nonlinear energy estimates ("Friedrichs-type") controlling high derivatives and in deriving high–frequency bounds.

The Evans function; spectral stability assumption

The linearized equations of (2) about the layer $\bar U$ are

$$U_t = LU := \sum_{j,k}(B^{jk}U_{x_k})_{x_j} - \sum_j (A^jU)_{x_j} \tag{5}$$

with initial data $U(0) = U_0$. Let $L_{\tilde\xi}$ be the Fourier transform of linearized operator L in transversal variables $\tilde x$ and write the eigenvalue equations $\lambda U - L_{\tilde\xi}U = 0$ in the first order ODEs

$$\begin{cases} W_{x_1} = A(x_1,\tilde\xi,\lambda)W \\ \Gamma W = 0 \qquad \text{on } \{x_1 = 0\}. \end{cases} \tag{6}$$

- The Evans function $D(\tilde\xi,\lambda)$, a Wronskian determinant of eigenvalue ODEs, is defined by

  $$D(\tilde\xi,\lambda) := \det(\ker\Gamma, W_1^+,\cdots,W_k^+)_{|_{x_1=0}}$$

  W_j^+'s span subspaces of solutions of (6) decaying at $x_1 = +\infty$.
- Zeros of the Evans function are eigenvalues. Thus, a necessary condition for stability: $D(\tilde\xi,\lambda)$ nonzero in $\{\Re e\lambda > 0\}$. We establish a sufficient condition as *strong spectral stability:*

  $$|D(\tilde\xi,\lambda)| \geq \theta(C) > 0 \tag{D}$$

  for $(\tilde\xi,\lambda)$ on bounded subsets $C\subset\{\tilde\xi\in\mathbb{R}^{d-1},\ \Re\lambda\geq 0\}\setminus\{0\}$.

Theorem 1, [NZ.2]: Multi-dimensional stability.

Assuming strong spectral stability (D), boundary layers $\bar U = (\bar\rho,\bar\rho\bar u,\bar\rho\bar E)^T$ with arbitrary amplitudes are (linearly and) nonlinearly stable in dimensions $d\geq 2$, with rates of decay, for s large and any $2\leq p\leq\infty$

$$\begin{aligned} |\tilde U(t) - \bar U|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p}(|U_0|_{L^1\cap H^s} + E_0) \\ |\tilde U(t) - \bar U|_{H^s} &\leq C(1+t)^{-\frac{d-1}{4}}(|U_0|_{L^1\cap H^s} + E_0), \end{aligned} \tag{7}$$

provided that the initial data perturbations $U_0 := \tilde U_0 - \bar U$ are sufficiently small in $L^1\cap H^s$ (and boundary perturbations).

Theorem 2, [NZ.1]: One-dimensional stability.

Boundary layers $\bar U$ are (linearly and) nonlinearly stable in $L^p\cap H^4$, $p>1$, with respect to small perturbations $U_0\in H^4$, if and only if strong spectral stability (nonvanishing of $D(\lambda)$ in $\{\Re e\lambda\geq 0\}\setminus\{0\}$) holds. In case of stability, we obtain the sharp rates of decay

$$\begin{aligned} \|\tilde U(x,t) - \bar U(x)\|_{L^p} &\leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad 1\leq p\leq\infty, \\ \|\tilde U(x,t) - \bar U(x)\|_{H^4} &\leq CE_0(1+t)^{-\frac{1}{4}}. \end{aligned}$$

(An extension of Yarahmadian and Zumbrun's result for strictly parabolic case).

Duhamel principle

We write the nonlinear perturbation equations in perturbation variable $U(x,t) := \tilde U(x,t) - \bar U(x)$

$$U_t - LU = \sum_j Q^j(U,U_x)_{x_j}, \tag{8}$$

where $Q^j(U,U_x) = \mathcal{O}(|U||U_x| + |U|^2)$. Then Duhamel principle gives

$$U(x,t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t-s)\partial_{x_j}Q^j(U,U_x)ds + \Gamma U(0,\tilde x,t) \tag{9}$$

where $\mathcal{S}(t) = e^{Lt}$ solution operator.

Decomposition of the solution operator

Inverse Laplace-Fourier transform:

$$\mathcal{S}(t)f := \frac{1}{(2\pi i)^d}\int_{\mathbb{R}^{d-1}_{\tilde\xi}}\int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t + i\tilde\xi\cdot\tilde x}(L_{\tilde\xi} - \lambda)^{-1}\hat f(x_1,\tilde\xi)d\lambda d\tilde\xi$$

Decompose the solution operator into $\mathcal{S}(t) := \mathcal{S}_{LF}(t) + \mathcal{S}_{HF}(t)$ where $\mathcal{S}_{HF}(t)$ corresponds to the integration over

$$\{\lambda\ :\ \Re e\lambda = -\theta_1\}\bigcap\{|\lambda| + |\tilde\xi|\geq r\},$$

and $\mathcal{S}_{LF}(t)$ corresponds to

$$\Gamma^{\tilde\xi} := \{\lambda\ :\ \Re e\lambda = -\theta_1(|\tilde\xi|^2 + |\Im m\lambda|^2)\}\bigcap\{|\lambda| + |\tilde\xi|\leq r\},$$

for $r,\theta_1 > 0$ sufficiently small.

Linearized estimates

High–frequency:

$$|\mathcal{S}_{HF}(t)f|_{L^2_x}\leq Ce^{-\theta_1 t}|f|_{H^3_x},$$

derived by applying a Laplace–Fourier transformed version of nonlinear (Goodman + Kawashima type) energy estimates to get an estimate for $(\lambda - L_{\tilde\xi})^{-1}$.

Low–frequency:

$$|\mathcal{S}_1(t)\partial_x^\beta f|_{L^2_x}\leq C(1+t)^{-(d-1)/4-|\beta|/2}|f|_{L^1_x} + C\beta_1(1+t)^{-(d-1)/4}|f|_{L^{1,\infty}_{\tilde x,x_1}}.$$

Main ingredients: Pointwise Green function approach, that is, to use the Evans function framework to construct and derive pointwise estimates on the resolvent kernel $G_{\tilde\xi,\lambda}(x_1,y_1)$.

An alternative [N.2]: Kreiss' symmetrizers-type analysis, that is, to directly estimate $(L_{\tilde\xi} - \lambda)^{-1}\hat f$ by using Guès-Métivier-Williams-Zumbrun's construction of Kreiss' symmetrizers [GMWZ6], which incidently extends the result to certain magnetohydrodynamics (MHD) boundary layers and allows us to drop a technical assumption on the so–called glancing set.

On verification of spectral stability

Analytical verifications: For small-amplitude layers, [GMWZ5] verifies the uniform Evans condition (D) in all dimensions $d\geq 2$. In the one–dimensional problem, the assumption can be verified via Kawashima-type energy estimates (still for small amplitudes).

Numerical verifications: In [CHNZ], we verify the spectral stability of arbitrary-amplitudes compressive boundary-layers for the isentropic gas.

References

[CHNZ] N. Costanzino, J. Humpherys, T. Nguyen, and K. Zumbrun, *Spectral stability of noncharacteristic boundary layers of isentropic Navier–Stokes equations*, ARMA, to appear.

[GMWZ5] O. Guès, G. Métivier, M. Williams, and K. Zumbrun. *Existence and stability of noncharacteristic hyperbolic-parabolic boundary-layers.* Preprint, 2008.

[N.2] T. Nguyen. *On asymptotic stability of noncharacteristic viscous boundary layers.* Preprint, 2009.

[NZ.1] T. Nguyen and K. Zumbrun. *Long-time stability of large-amplitude noncharacteristic boundary layers for hyperbolic-parabolic systems.* Preprint, 2008.

[NZ.2] T. Nguyen and K. Zumbrun. *Long-time stability of multi-dimensional noncharacteristic viscous boundary layers.* Preprint, 2008.

Acknowledgements: Thanks the Institute for Mathematics and Its Applications for their generous support. This work was supported in part by the National Science Foundation award number DMS-0300487.