A New Fourth-Order Non-Oscillatory Central Scheme For Hyperbolic Conservation Laws

Arshad A I Peer
arshad.peer@umail.uom.ac.mu
Department of Mathematics
University of Mauritius

Abstract

We propose a new fourth-order non-oscillatory central scheme for computing approximate solutions of hyperbolic conservation laws. A piecewise cubic polynomial is used for the spatial reconstruction and for the numerical derivatives we choose genuinely fourth-order accurate non-oscillatory approximations. The solution is advanced in time using natural continuous extension of Runge-Kutta methods. Numerical tests on both scalar and gas dynamics problems confirm that the new scheme is non-oscillatory and yields sharp results when solving profiles with discontinuities. Experiments on nonlinear Burgers equation indicate that our scheme is superior to existing fourth-order central schemes in the sense that the total variation (TV) of the computed solutions are closer to the total variation of the exact solution.

Introduction

We propose a new fourth-order central scheme for hyperbolic conservation laws

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^d,\ d \geq 1, \tag{1}$$

subject to the initial conditions $u(x, t = 0) = u_0(x)$. Following the NT scheme (Nessyahu & Tadmor, JCP, 1990), we derive a scheme which employs a non-oscillatory reconstruction by combining a higher-order polynomial with a mechanism to eliminate spurious oscillations. Assuming that the cell averages $\bar{u}_j^n$ at time t^n are known, we reconstruct piecewise cubic polynomials $R_j^n(x)$. Integrating (1) over $[x_j,\, x_{j+1}] \times [t^n,\, t^{n+1}]$, and using Simpson's quadrature formula for the time integral, we get

$$\begin{aligned}\bar{u}_{j+\frac{1}{2}}^{n+1} =& \frac{1}{h}\left[\int_{x_j}^{x_{j+\frac{1}{2}}} R_j^n(x)\,dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} R_{j+1}^n(x)\,dx\right] \\ &+ \lambda \sum_{l=0}^{2} \gamma_l \left[f\left(u(x_j,\, t^n + \beta_l \Delta t)\right) - f\left(u(x_{j+1},\, t^n + \beta_l \Delta t)\right)\right],\end{aligned}$$

where $\lambda = \Delta t/h$, γ_l and β_l are the weights and nodes of the quadrature formula, and $u(x_j,\, t^n + \beta_l \Delta t)$ are the intermediate values.

A New Non-Oscillatory Reconstruction

We choose the interpolating polynomial $R_j^n(x)$ to have the form

$$R_j^n(x) = u_j^n + u_j'\left(\frac{x - x_j}{h}\right) + \frac{1}{2!}u_j''\left(\frac{x - x_j}{h}\right)^2 + \frac{1}{3!}u_j'''\left(\frac{x - x_j}{h}\right)^3,$$

where $R_j^n(x)$ obeys the conservation property $\frac{1}{h}\int_{I_j} R_j^n(x)\,dx = \bar{u}_j^n$, that is, u_j^n must satisfy

$$u_j^n = \bar{u}_j^n - \frac{1}{24}u_j''. \tag{2}$$

We require the numerical derivatives $\frac{1}{h}u_j'$, $\frac{1}{h^2}u_j''$ and $\frac{1}{h^3}u_j'''$ to be fully non-oscillatory as in the NT scheme compared to ENO reconstructions. We employ the fourth-order accurate modified UNO MinMod limiters of Peer *et al.* (CMMSE, 2005). u_j''' depends on its two neighbouring third-order differences

$$u_j''' = \text{MM}\left(\Delta^3\bar{u}_{j-\frac{1}{2}}, \Delta^3\bar{u}_{j+\frac{1}{2}}\right),$$

where $\Delta^3\bar{u}_{j+\frac{1}{2}} = \Delta^2\bar{u}_{j+1} - \Delta^2\bar{u}_j$. Similar to the UNO limiter, u_j' combines higher-order terms in smooth regions to attain high-order accuracy

$$\begin{aligned}u_j' =& \text{MM}\left(\Delta\bar{u}_{j-\frac{1}{2}} + \frac{1}{2}\text{MM}\left(\Delta^2\bar{u}_{j-1} + \frac{7}{12}u_{j-1}''', \Delta^2\bar{u}_j - \frac{5}{12}u_j'''\right),\right. \\ &\left.\Delta\bar{u}_{j+\frac{1}{2}} - \frac{1}{2}\text{MM}\left(\Delta^2\bar{u}_j + \nu_3 = \frac{5}{12}u_j''', \Delta^2\bar{u}_{j+1} - \frac{7}{12}u_{j+1}'''\right)\right).\end{aligned}$$

This mechanism is aimed at removing spurious oscillations allowed by other reconstructions like ENO and its weighted versions. In case extremas cannot be avoided, the accuracy of the limiters decreases until non-oscillatory approximations are obtained.

The Reconstruction of Point-Values

We approximate the point-value u_j^n of (2) from cell averages, while looking for high accuracy and avoiding oscillations:

$$u_j'' = \text{MM}\left(\Delta^2\bar{u}_{j-1}^n + u_{j-1}''',\ \Delta^2\bar{u}_j^n,\ \Delta^2\bar{u}_{j+1}^n - u_{j+1}'''\right),$$

where the MM limiter with three arguments can be written in the form

$$\begin{aligned}\text{MM}(x_1,\ x_2,\ x_3) =& \frac{1}{4}(\text{sign}(x_1) + \text{sign}(x_2) + \text{sign}(x_3) + \text{sign}(x_1\, x_2\, x_3)) \\ &\times \min(|x_1|, |x_2|, |x_3|).\end{aligned}$$

We obtain the intermediate time values of the quadrature formula from the fourth-order Natural Continuous Extension of Runge-Kutta schemes. This require the non-oscillatory first derivative of fluxes:

$$\begin{aligned}f_j' = \text{MM}&\left(\Delta f_{j-\frac{1}{2}} + \tfrac{1}{2}\text{MM}\left(\Delta^2 f_{j-1} + \tfrac{2}{3}f_{j-1}''', \Delta^2 f_j - \tfrac{1}{3}f_j'''\right),\right. \\ &\left.\Delta f_{j+\frac{1}{2}} - \tfrac{1}{2}\text{MM}\left(\Delta^2 f_j + \tfrac{1}{3}f_j''', \Delta^2 f - \tfrac{2}{3}f_{j+1}'''\right)\right),\end{aligned}$$

where

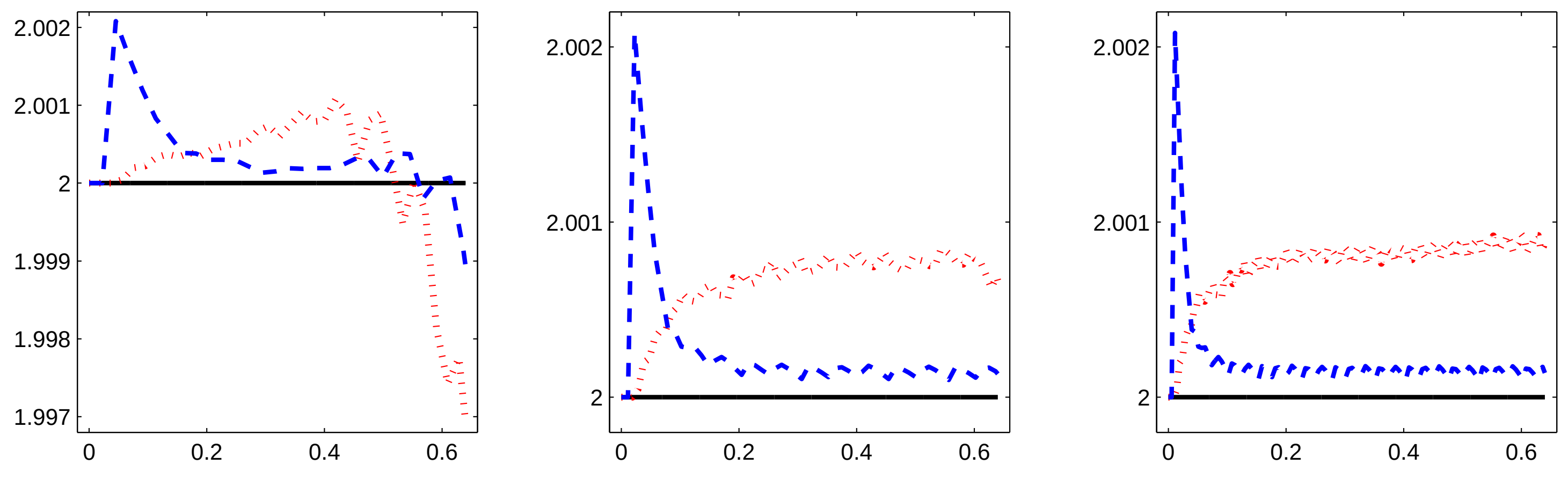
$$f_j''' = \text{MM}\left(\Delta^3 f_{j-\frac{1}{2}}, \Delta^3 f_{j+\frac{1}{2}}\right).$$

Numerical Experiments

We call the new scheme CNO4. A linear stability analysis gives the critical Courant number $\lambda_{\max} = 0.3408$. Below we compare our results with CWENO of Levy *et al.* (M2AN, 1999).

Burgers' Equation

$u_t + (0.5\,u^2)_x = 0$, where $u(x,\,0) = 1$ for $|x| < 1/3$ and $u(x,\,0) = 0$ elsewhere. $T = 0.64$ and $\lambda = \frac{2}{3}\,\lambda_{\max}$.

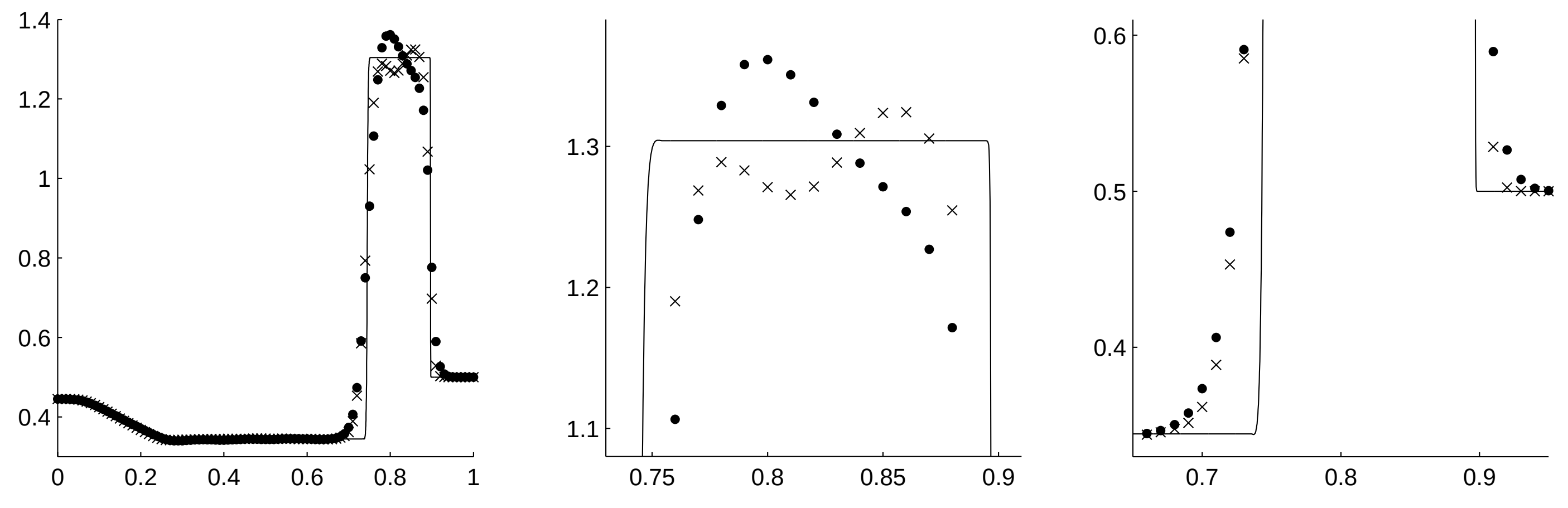


(a) $N = 40$ (b) $N = 80$ (c) $N = 160$

Figure 1: TV of approximations (—: Exact, - -: CNO4, · · ·: CWENO).

Lax Problem

(Lax, CPAM. 1954) $T = 0.16$ and $N = 100$.

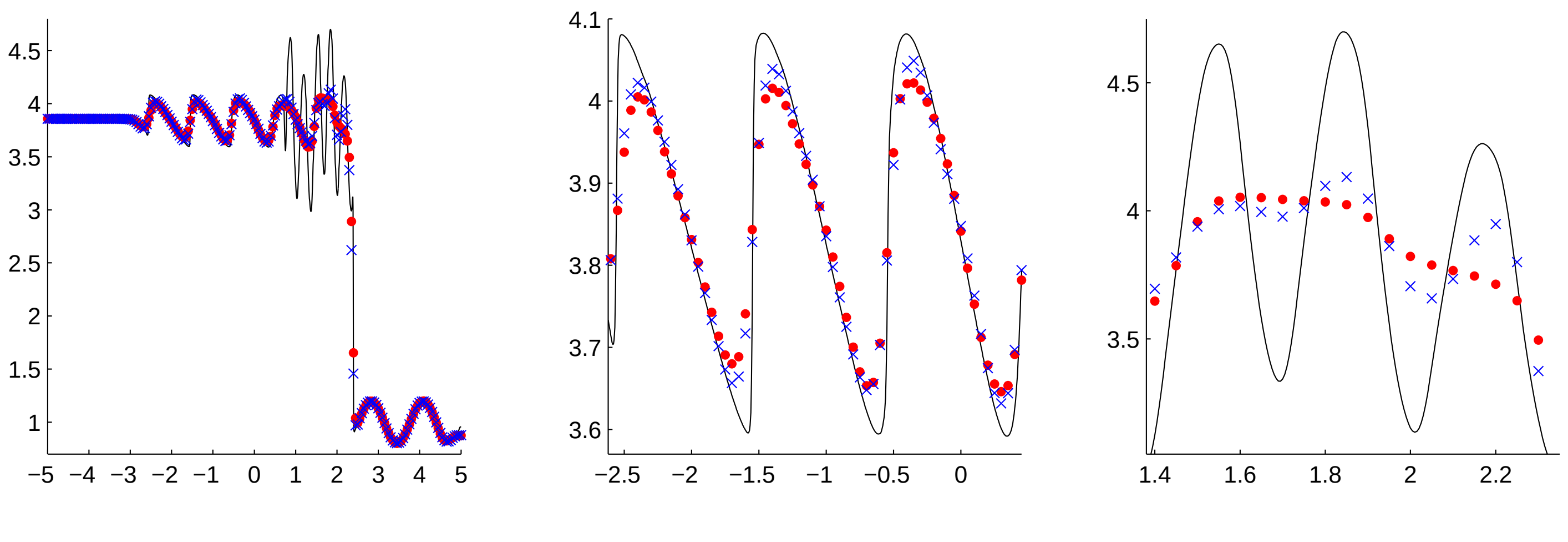


(a) Density (b) Zoomed region (c) Zoomed region

Figure 2: Density profile by CNO4 "×" and CWENO "·".

Shock-Entropy Test

(Shu & Osher, JCP. 1989) $T = 1.8$ and $N = 200$.



(a) Density (b) Zoomed region (c) Zoomed region

Figure 3: Density profile by CNO4 "×" and CWENO "·".

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