A slip-inflow boundary value problem for a steady flow of compressible fluid in a cylindrical domain

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Abstract

We investigate a steady flow of a viscous, compressible fluid with inflow boundary condition on the density and inhomogeneous slip boundary conditions on the velocity in a cylindrical domain $\Omega = \Omega_0 \times (0, L) \in \mathbb{R}^3$. We show existence of a strong solution $(v, \rho) \in W^2_p(\Omega) \times W^1_p(\Omega)$, $p > 3$, where v is the velocity of the fluid and ρ is the density, that is a small perturbation of a constant flow ($\bar{v} \equiv [1, 0, 0], \bar{\rho} \equiv 1$). We also show that this solution is unique in a class of small perturbations of $(\bar{v}, \bar{\rho})$. In order to show existence of the solution we construct a sequence (v^n, ρ^n) that is bounded in $W^2_p(\Omega) \times W^1_p(\Omega)$ and satisfies the Cauchy condition in $H^1(\Omega) \times L_\infty(0, L, L_2(\Omega_0))$, what enables us to deduce that the weak limit of a subsequence of (v^n, ρ^n) is in fact a strong solution to our problem.

1. Introduction and main results

The mathematical description of a flow of viscous compressible fluids usually lead to problems of a mixed character as the momentum equation is elliptic (or parabolic) in the velocity while the continuity equation is hyperbolic in the density. Therefore, if the velocity does not vanish on the boundary we have to prescribe the density on the part where the flow enters the domain, we shall call this part the inflow part, and a singularity arises at the junction of the inflow part and the remaining part of the boundary.

So far, all known existence results for strong solutions are subject to constraints either on the smallness of the data or the geometry of the boundary, and the problems are usually considered with homogeneous Dirichlet boundary condition. Some smallness is also necessary for our result, and on the boundary we prescribe inhomogeneous slip boundary conditions that allow to describe precisely the action between the fluid and the boundary. More precisely, the flow is governed by the system

$$\begin{cases} \rho v \cdot \nabla v - \mu \Delta v - (\mu + \nu) \nabla \operatorname{div} v + \nabla \pi(\rho) = 0 & \text{in} \quad \Omega, \\ \operatorname{div}(\rho v) = 0 & \text{in} \quad \Omega, \\ n \cdot \mathbf{T}(v, \pi(\rho)) \cdot \tau_k + f v \cdot \tau_k = b_k, \quad k = 1, 2 & \text{on} \quad \Gamma, \\ n \cdot v = d & \text{on} \quad \Gamma, \\ \rho = \rho_{in} & \text{on} \quad \Gamma_{in}, \end{cases} \tag{1}$$

where $v : \mathbb{R}^3 \to \mathbb{R}^3$ is the unknown velocity field of the fluid and $\rho : \mathbb{R}^3 \to \mathbb{R}$ is the unknown density. The pressure $\pi(\rho)$ is a C^3-function, μ and ν are viscisity coefficients and $f > 0$ is a friction coefficient. The domain $\Omega = \Omega_0 \times (0, L)$ where $\Omega_0 \in \mathbb{R}^2$ is a set with a boundary regular enough and L is a positive constant.

We want to show existence of a solution that can be considered a small perturbation of a constant flow $(\bar{v}, \bar{\rho}) \equiv ([1, 0, 0], 1)$. Thus we denote the subsets of the boundary $\Gamma = \partial\Omega$ as $\Gamma = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_0$, where $\Gamma_{in} = \{x \in \Gamma : \bar{v} \cdot n < 0\}$, $\Gamma_{out} = \{x \in \Gamma : \bar{v} \cdot n > 0\}$ and $\Gamma_0 = \{x \in \Gamma : \bar{v} \cdot n = 0\}$ (see fig.1).

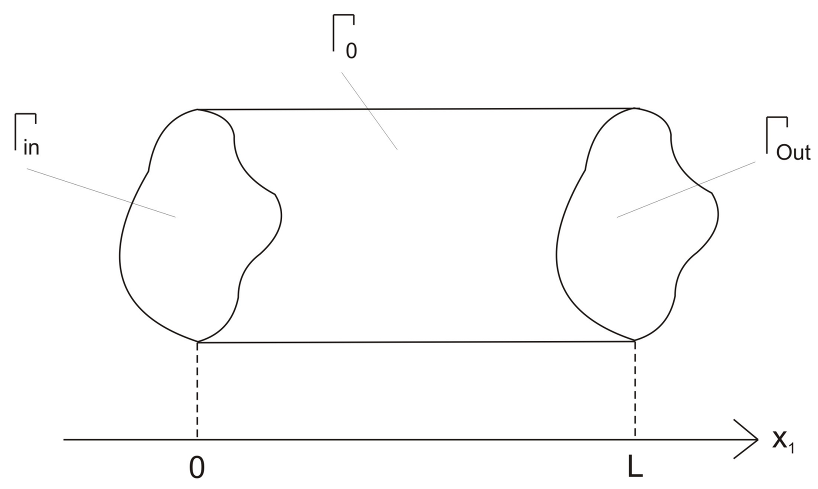

Figure 1: *The domain*

To formulate our main result in a convenient way we define a quantity that measures the distance of the boundary data from $(\bar{v}, \bar{\rho})$:

$$D_0 = ||b_i - f\tau_i^{(1)}||_{W_p^{1-1/p}(\Gamma)} + ||d - n^{(1)}||_{W_p^{2-1/p}(\Gamma)} + ||\rho_{in} - 1||_{W_p^1(\Gamma_{in})}. \tag{2}$$

Our main result is

Theorem 1 *Assume that D_0 defined in (2) is small enough, f is large enough and $p > 3$. Then there exists a solution $(v, \rho) \in W^2_p(\Omega) \times W^1_p(\Omega)$ to the system (1) and*

$$||v - \bar{v}||_{W_p^2(\Omega)} + ||\rho - \bar{\rho}||_{W_p^1(\Omega)} \le E(D_0), \tag{3}$$

where $E(D_0)$ can be arbitrarily small provided that D_0 is small enough. This solution in unique in the class of solutions satisfying the estimate (3).

The term $u \cdot \nabla w$ in the continuity equation makes it impossible to show the existence directly by a fixed point argument, thus we apply the method of successive approximations. The existence of the approximating sequence is subject to solvability of the linear system and the proof of convergence of the sequence of approximations is based on the estimates for the linear system. Therefore we first linearize the system and then show the aforementioned estimates. Next we solve the linear system using a modification of the Galerkin method by application of DiPerna-Lions theory of transport equation. Then we show that the sequence of approximations is bounded in $W^2_p(\Omega) \times W^1_p(\Omega)$ and satisfy the Cauchy condition in $L_\infty(0, L; L_2(\Omega))$. These estimates enable us to show the convergence of the approximating sequence to the solution of the system. Finally we show that the solution is unique in a class of solutions satisfying (3).

2. Linarization and *a priori* bounds

We reformulate the problem taking the perturbations of $(\bar{v}, \bar{\rho})$ as unknown functions. More precisely, we construct u_0, an extension of the boundary data $n \cdot u_0|_\Gamma = d - n^{(1)}$ and consider

$$u = v - \bar{v} - u_0 \quad \text{and} \quad w = \rho - \bar{\rho}.$$

The couple (u, w) satisfies

$$\begin{cases} \partial_{x_1} u - \mu \Delta u - (\nu + \mu) \nabla \operatorname{div} u + \pi'(1) \nabla w = F(u, w) & \text{in} \quad \Omega, \\ \operatorname{div} u + \partial_{x_1} w + (u + u_0) \cdot \nabla w = G(u, w) & \text{in} \quad \Omega, \\ n \cdot 2\mu \mathbf{D}(u) \cdot \tau_i + f\, u \cdot \tau_i = B_i, \quad i = 1, 2 & \text{on} \quad \Gamma, \\ n \cdot u = 0 & \text{on} \quad \Gamma, \\ w = w_{in} & \text{on} \quad \Gamma_{in}. \end{cases} \tag{4}$$

The precise form of the functions F and G is not important, what matters is that they satisfy

$$||F(u, w)||_{L_p} + ||G(u, w)||_{W_p^1} \le C\,[(||u||_{W_p^2} + ||w||_{W_p^1})^2 + ||u_0||_{W_p^2} + ||B||_{W_p^{1-1/p}(\Gamma)}]. \tag{5}$$

Defition of approximating sequence and linearization. In order to prove Theorem 1 it is enough to show existence of solution to (4) under some smallness assumptions on B and u_0. To this end we define the approximations of the solution as a sequence of solutions to the systems

$$\begin{cases} \partial_{x_1} u^{n+1} - \mu \Delta u^{n+1} - (\nu + \mu) \nabla \operatorname{div} u^{n+1} + \pi'(1) \nabla w^{n+1} = F(u^n, w^n) & \text{in} \quad \Omega, \\ \operatorname{div} u^{n+1} + \partial_{x_1} w^{n+1} + (u^n + u_0) \cdot \nabla w^{n+1} = G(u^n, w^n) & \text{in} \quad \Omega, \end{cases} \tag{6}$$

supplied with the boundary conditions $(4)_{3,4,5}$.

Definition 1 *By linearization of (4) we mean the system (4) with given functions $F \in L_p(\Omega)$ and $G \in W^1_p(\Omega)$ on the r.h.s. and the term $(u + u_0) \cdot \nabla w$ replaced by $(\bar{u} + u_0)$, where $\bar{u} \in W^2_p(\Omega)$ is a given function satisfying $\bar{u} \cdot n|_\Gamma = 0$.*

Bounds for the linear system. Multiplying $(4)_1$ by u, integrating over Ω and then applying (11) we get the energy estimate for the linear system:

Lemma 1 *Let (u, w) be a solution to the linearization of (4). Then*

$$||u||_{H^1} + ||w||_{L_\infty(0,L;L_2(\Omega_0))} \le C\,[||F||_{V^*} + ||G||_{L_2} + ||B||_{L_2(\Gamma)} + ||w_{in}||_{L_2(\Gamma_{in})}], \tag{7}$$

where

$$V = \{v \in H^1(\Omega) : v \cdot n|_\Gamma = 0\} \tag{8}$$

and V^* is the dual space of V.

Next we show that the vorticity of the velocity $\alpha = \operatorname{curl} u$ satisfies

$$\begin{cases} \partial_{x_1} \alpha - \mu \Delta \alpha = \operatorname{curl} F & \text{in} \quad \Omega, \\ \alpha \cdot \tau_2 = (2\chi_1 - \frac{f}{\nu}) u \cdot \tau_1 + \frac{B_1}{\nu} & \text{on} \quad \Gamma, \\ \alpha \cdot \tau_1 = (\frac{f}{\nu} - 2\chi_2) u \cdot \tau_2 - \frac{B_2}{\nu} & \text{on} \quad \Gamma, \\ \operatorname{div} \alpha = 0 & \text{on} \quad \Gamma. \end{cases}$$

The classical theory of elliptic equations applied to this system together with the energy estimate (7) yields

$$||\alpha||_{W_p^1} \le C(\epsilon)\,[||F||_{L_p} + ||G||_{W_p^1} + ||w_{in}||_{L_2(\Gamma_{in})} + ||u||_{W_p^{1-1/p}(\Gamma)} + ||B||_{W_p^{1-1/p}(\Gamma)}] + \epsilon ||u||_{W_p^2} \quad \forall\, \epsilon > 0.$$

Next, applying the Helmholtz decomposition to momentum equation we show that the density satifies the transport equation

$$\bar{\gamma} w + w_{x_1} + (\bar{u} + u_0) \nabla w = H, \tag{9}$$

where $H \in W^1_p$ depends on w and u and we have the bound on ∇H. From the equation (9) we derive the estimate

$$||w||_{W_p^1} \le C\,\big[||H||_{W_p^1} + ||w_{in}||_{W_p^1(\Gamma_{in})}\big],$$

which together with the regularity of the Lame system let us conclude the estimate in $W^2_p(\Omega) \times W^1_p(\Omega)$:

Lemma 2 *If (u, w) is a solution to the linearization of (4) then*

$$||u||_{W_p^2} + ||w||_{W_p^1} \le C\,[||F||_{L_p} + ||G||_{W_p^1} + ||B||_{W_p^{1-1/p}(\Gamma)} + ||w_{in}||_{W_p^1(\Gamma_{in})}]. \tag{10}$$

3. Solution of the linear system

First we show the existence of a weak solution using the Galerkin method. A standard Galerkin approach however does not work due to the mixed character of the problem, hence we modify it in a way that enables to deal with the continuity equation. Namely, the finite dimensional approximations to the velocity are searched for in a standard form $u^N = \sum c_i^N \phi_i$ where $\{\phi_i\}$ is the basis of the space V defined in (8), and the approximations of the density are searched for as $w^N = S(G - \operatorname{div} u^N)$ where $S : L_2(\Omega) \to L_\infty(0, L; L_2(\Omega_0))$ is defined as

$$w = S(v) \iff \begin{cases} \partial_{x_1} w + \bar{u} \cdot \nabla w = v & \text{in} \quad \mathcal{D}'(\Omega), \\ w = w_{in} & \text{on} \quad \Gamma_{in}. \end{cases}$$

The above equation can be solved by the method of characteristics for a smooth function v with an estimate

$$||S(v)||_{L_\infty(0,L;L_2(\Omega_0))} \le C\,[||w_{in}||_{L_2(\Gamma_{in})} + ||v||_{L_2(\Omega)}], \tag{11}$$

which let us extend S to $L_2(\Omega)$ by a density argument.

We show the existence of solutions (u^N, w^N) to finite dimensional problems and next show the convergence to the weak solution of the linear problem, let us denote it (u_l, w_l). Finally using the estimate in $W^2_p(\Omega) \times W^1_p(\Omega)$ and the symmetry of slip boundary conditions we show that (u_l, w_l) is a strong solution to the linear problem.

4. Proof of Theorem 1

Bounds on the sequence of approximations. The solution of the linear system gives existence of the approximating sequence defined in (6). Using the esimate (10) together with (5) we show

Lemma 3 *Let $\{(u^n, w^n)\}$ be a sequence of solutions to (6) starting from $(u^0, w^0) = (0, 0)$. Then*

$$||u^n||_{W_p^2} + ||w^n||_{W_p^1} \le M, \tag{12}$$

where M can be arbitrarily small provided that u_0 and B are small enough.

With this result we subtract the equations on u^{n+1} and u^{m+1} and show that

Lemma 4 *Let $\{(u^k, w^k)\}$ be a sequence of solutions to (6) with $(u^0, w^0) = (0, 0)$. Then we have*

$$||u^{n+1} - u^{m+1}||_{H^1} + ||w^{n+1} - w^{m+1}||_{L_\infty(0,L;L_2(\Omega_0))} \le E(M)\,\big(||u^n - u^m||_{H^1} + ||w^n - w^m||_{L_\infty(0,L;L_2(\Omega_0))}\big),$$

where M is the constant from (12).

This result clearly implies the Cauchy condition for (u^n, w^n) in $H^1(\Omega) \times L_\infty(0, L; L_2(\Omega_0))$.

Existence. The convergence of (u^n, w^n) in $H^1(\Omega) \times L_\infty(0, L; L_2(\Omega_0))$ supplied with (12) let us pass to the limit in the weak formulation of (4). We hence obtain (u, w), a weak solution to (4). On the other hand, $(u^n, w^n) \rightharpoonup (u, w)$ in $W^2_p(\Omega) \times W^1_p(\Omega)$ and we conclude that (u, w) is a strong solution.

Uniqueness. In order to show the uniqueness we come back to (1) and consider two solutions $(v_1, \rho_1), (v_2, \rho_2)$. Subtracting the equations on (v_1, ρ_1) and (v_2, ρ_2) we get a system on the difference. To this system we apply a modification of the proof of the energy estimate (7) to obtain, after direct but quite lengthy computations,

$$||v_1 - v_2||^2_{H^1} + ||\rho_1 - \rho_2||^2_{L_2} = 0,$$

what completes the proof.

References

[1] T.Piasecki, *On an inhomogemeous slip boundary value problem for a steady flow of viscous compressible fluid in a cylindrical domain*, arxiv: 0907.0716