On a model of granular flow

Wen Shen, Penn State University, U.S.A.
Joint work with Debora Amadori, University of L'Aquila, Italy

References:

[1] Wen Shen, On the Shape of Avalanches, *J. Math. Anal. Appl.*, **339**(2008), 828-838.

[2] Debora Amadori and Wen Shen, Global Existence of large BV solutions for a Model of Granular Flow, To appear in Communications in Partial Differential Equations.

[3] Debora Amadori and Wen Shen, The Slow Erosion Limit for a Model of Granular Flow, preprint 2008.

Available online: www.math.psu.edu/shen_w/Papers.html
Norwegian Preprint Server: www.math.ntnu.no/conservation/

The model for granular matter (Hadeler and Kuttler, *Granular Matter*, 1999):

$$\begin{cases} h_t = \operatorname{div}(h\nabla u) - (1-|\nabla u|)h \\ u_t = (1-|\nabla u|)h \end{cases}$$

h: thickness of the moving layer

u: the height of the standing layer

moving layer moves down-hill

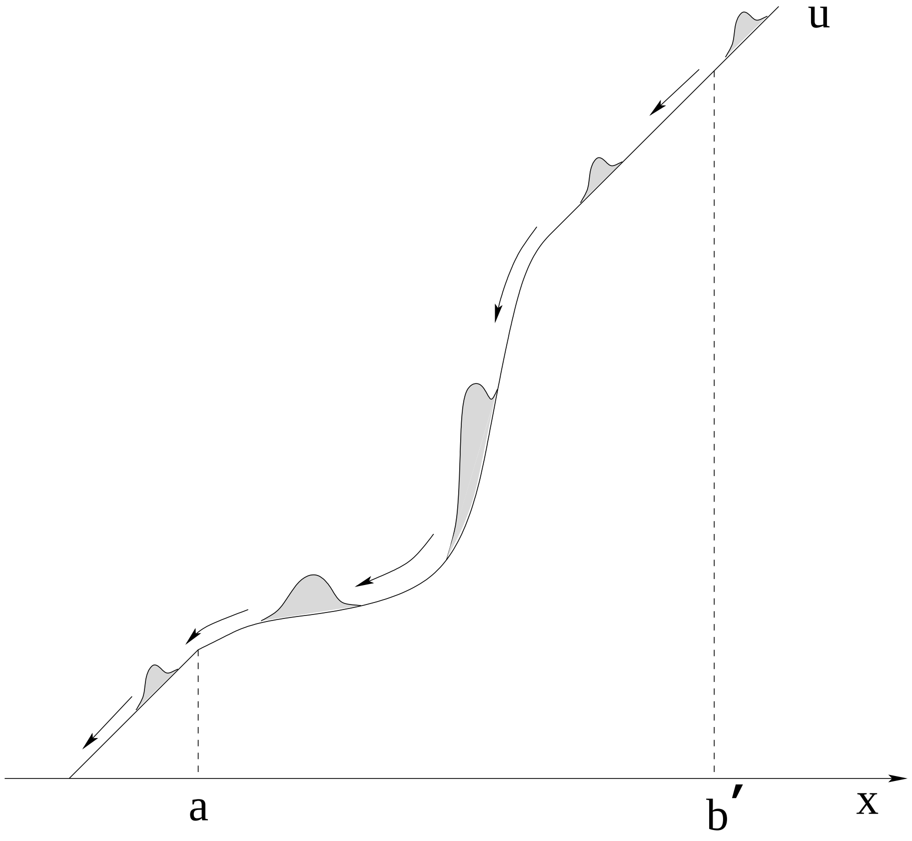
in steepest descent direction

speed equals to the slope

erosion if $|\nabla u| > 1$

deposit if $|\nabla u| < 1$

critical slope: $|\nabla u| = 1$



A 1D model:

Assume the slope $p = u_x$ doesn't change sign, say $p \geq 0$, we get a system of balance laws

$$\begin{cases} h_t - (hp)_x &= (p-1)h \\ p_t + ((p-1)h)_x &= 0 \end{cases} \tag{1}$$

Jacobian matrix:

$$A(h,p) = \begin{pmatrix} -p & -h \\ p-1 & h \end{pmatrix}$$

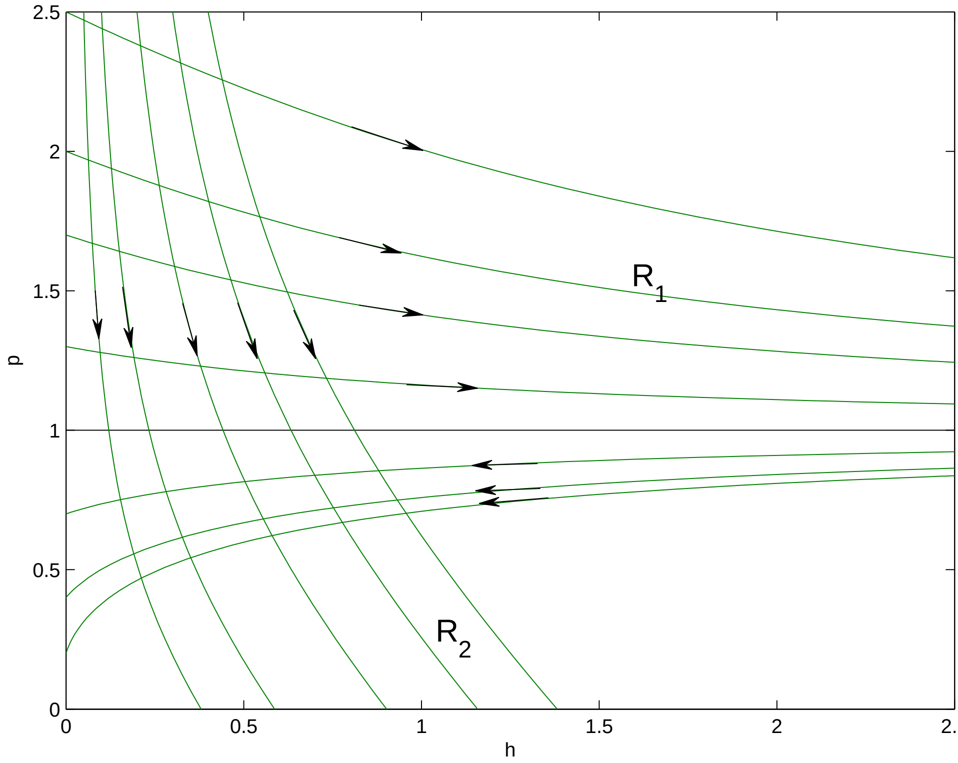
eigenvalues:

$$\lambda_{1,2} = \frac{h - p \pm \sqrt{(p-h)^2 + 4h}}{2}$$

for small $h \approx 0$, one has

$$\lambda_1 = -p + \frac{p-1}{p}h + \mathcal{O}(h^2), \qquad \lambda_2 = \frac{h}{p} + \mathcal{O}(h^2)$$

Integral curves:



$$r_1 \bullet \lambda_1 \approx \frac{(p-1)}{p}$$

$$r_2 \bullet \lambda_2 \approx -\frac{h}{p^2}$$

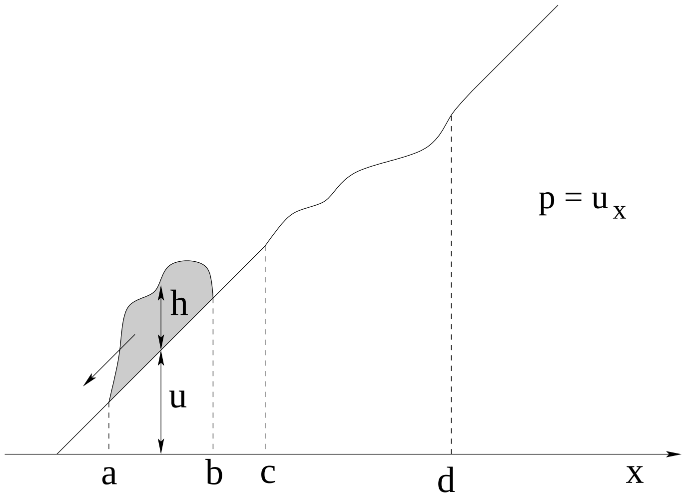
1st field is genuinely nonlinear away from the line $p = 1$

2nd field is genuinely nonlinear away from the line $h = 0$

The system: weakly linearly degenerate at $(h, p) = (0, 1)$.

Global existence of smooth solutions: (W.S., JMAA 2008)

Decoupled initial data:



$$h(0,x) = \phi(x), \qquad p(0,x) = 1 + \psi(x)$$

where $\text{supp}(\phi) \in [a,b]$,
$\text{supp}(\psi) \in [c,d]$
and $a < b < c < d$.

explicit solutions: $\qquad h(t,x) = \phi(x+t) \qquad p(t,x) = 1 + \psi(x)$

We study stability of such solutions.

Perturbations: $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ with supp in [a,d]
and $|\tilde{\phi}'(x)| \leq \delta\,, |\tilde{\psi}'(x)| \leq \delta$, ($\delta$ small)

initial data:
$h(0,x) = \phi(x) + \tilde{\phi}(x), \qquad p(0,x) = 1 + \psi(x) + \tilde{\psi}(x)$

Results:
– Cauchy problem gives decoupled solution in finite time.
– no shocks (discontinuities) developed
– global Lipschitz continuous solutions

Proof:
method of characteristics (Ta-Tsian Li, no source term):
bounds on the $\mathbf{L}^\infty$ and $\mathbf{L}^1$ norms of h_x and p_x.

Global existence of large BV solutions: (Amadori & Shen, to appear in CPDE 2009)

Assumptions on initial data $h(0,x) = \bar{h}(x)$, $p(0,x) = \bar{p}(x)$:

—- Tot.Var$\{\bar{h}\} \leq M$, Tot.Var$\{\bar{p}\} \leq M$

—- $\|\bar{h}\|_{\mathbf{L}^1} \leq M$, $\|\bar{p} - 1\|_{\mathbf{L}^1} \leq M$, $\bar{p}(x) \geq p_0 > 0$

—– $\|\bar{h}\|_{\mathbf{L}^\infty} \leq \delta$, for small $\delta > 0$

Result: Global existence of large BV solutions
For a sufficiently small $\delta > 0$, our Cauchy problem has a unique entropy weak solution, for all $t \geq 0$, with uniformly bounded total variation.

Novelty:
large BV data + large L^∞ norm + source term + 1^{st} field is NOT "either genuinely nonlinear or linearly degenerate"

For 2×2 systems, one can measure wave strength in Riemann coordinates.

Standard interaction estimates are cubic:

$$\left|\sigma_1' - \sigma_1\right| + \left|\sigma_2' - \sigma_2\right| = \mathcal{O}(1) \cdot |\sigma_1| \cdot |\sigma_2| \cdot (|\sigma_1| + |\sigma_2|)$$

Known results with global BV (without source terms):

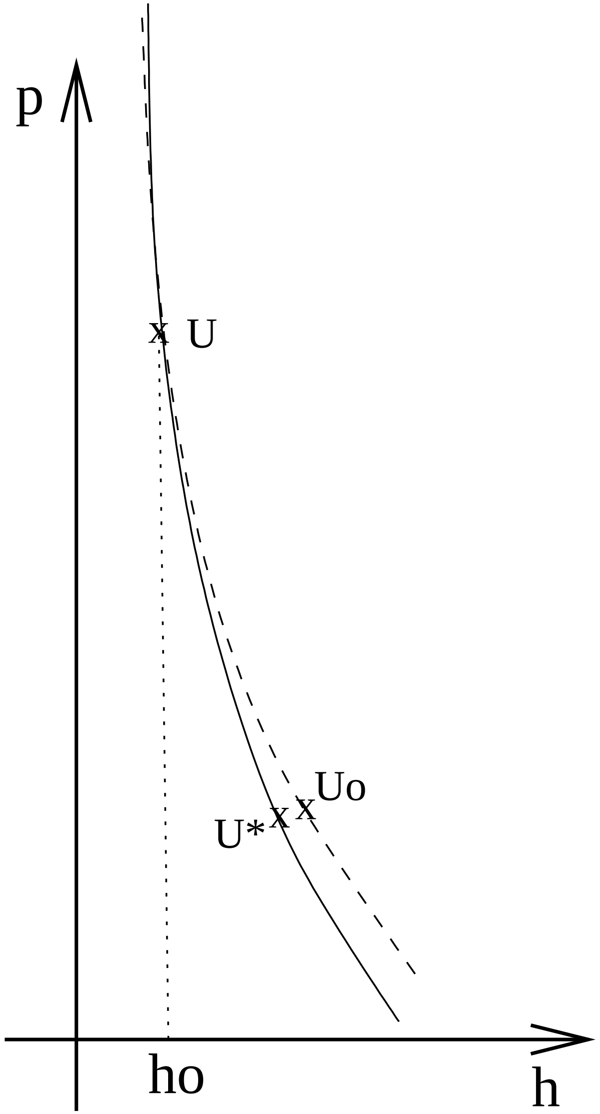
(1). For Temple class (shock & rarefaction curves coincide)

$$\left|\sigma_1' - \sigma_1\right| + \left|\sigma_2' - \sigma_2\right| = 0$$

⇒ global BV solutions

(2). If L^∞-norms of both components are small, GNL system
⇒ global BV solutions (Glimm & Lax, *AMS Memoir*, 1970)

What is special about our case:



– at $h = 0$, the 2-characteristic is straight line and linearly degenerate.

– so, if $h \approx 0$, we have

“almost-Temple class”:

Rarefaction curve and shock curve through the same point are very close to each other

$$|U^* - U_o| = \mathcal{O}(1) \cdot h_o^2$$

– wave speeds of different families are strictly separated

Our interaction estimates are much improved, for $h << 1$:

(1) 2 waves from diff families: $(\sigma_h, \sigma_p) \mapsto (\sigma'_h, \sigma'_p)$:

$$\left|\sigma'_h - \sigma_h\right| + \left|\sigma'_p - \sigma_p\right| = \mathcal{O}(1) \cdot \|h\|_\infty \cdot |\sigma_h| \cdot |\sigma_p|$$

(2) 2 waves from the 2nd family: $(\sigma_p, \tilde{\sigma}_p) \mapsto (\sigma'_h, \sigma'_p)$:

$$|\sigma'_h| + \left|\sigma'_p - (\sigma_p + \tilde{\sigma}_p)\right| = \mathcal{O}(1) \cdot \|h\|_\infty \cdot |\sigma_p| \cdot |\tilde{\sigma}_p|$$

(3) 2 waves from the 1st family: $(\sigma_h, \tilde{\sigma}_h) \mapsto (\sigma'_h, \sigma'_p)$:

$$\left|\sigma'_h - (\sigma_h + \tilde{\sigma}_h)\right| + |\sigma'_p| = \mathcal{O}(1) \cdot |p_l - 1| \left(|\sigma_h| + |\tilde{\sigma}_h|\right) \cdot |\sigma_h| \cdot |\tilde{\sigma}_h|$$

All interaction estimates contain an additional factor of $\|h\|_\infty$!
So: if $\|h\|_\infty$ remains small, then total amount of new waves produced by interaction remains small
$\Rightarrow$ can neglect the interactions and focus on the source term effect

Source terms: quadratic form $(p-1)h$

– mass of h flows with speed $-p$ (< 0)
– mass of $p-1$ moves with speed h (≥ 0)
So: two mass flows are strictly transversal

both masses h and $p-1$ have large, but bounded $\mathbf{L}^1$ norms

Total strength of the source term $= \ \mathcal{O}(1) \cdot \|h\|_{\mathbf{L}^1} \cdot \|p-1\|_{\mathbf{L}^1}$
large but bounded

In addition, since h itself is a factor in the source term, one can obtain a uniform bound on $\|h\|_{\mathbf{L}^\infty}$, valid for all times $t \geq 0$.

Sketch of the proof: construct approximate solutions (h^Δ, p^Δ) to (1) by operator splitting method.

(A). On each subinterval $[t_{k-1}\,,\ t_k[$ the functions (h^Δ, p^Δ) provide an approximate solution to the system of conservation laws

$$\begin{cases} h_t - (hp)_x &= 0\,, \\ p_t + ((p-1)h)_x &= 0\,, \end{cases}$$

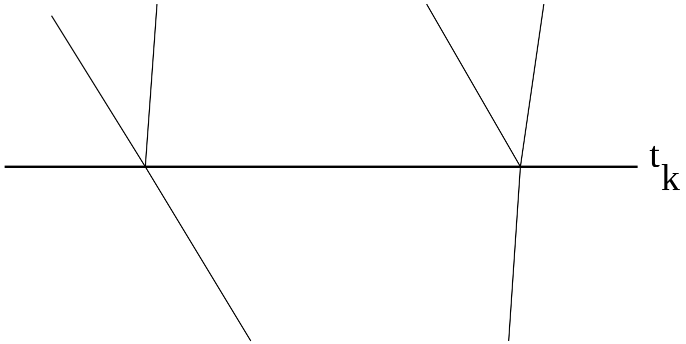
constructed by a wave-front tracking algorithm.

(B). Then, at each time t_k, take care of the source term

$$\begin{cases} h^\Delta(t_k) &= h^\Delta(t_k-) + \Delta t \left[p^\Delta(t_k-) - 1\right] h^\Delta(t_k-)\,, \\ p^\Delta(t_k) &= p^\Delta(t_k-)\,. \end{cases}$$

Time step estimates:

At a time when the source term is added, a wave front will split into 2 waves.



If the initial jump is of the first family, with strength σ_h, we have

$$\sigma_h \qquad \Longrightarrow \qquad \begin{cases} \sigma_h^+ &= \sigma_h + \mathcal{O}(1) \cdot \Delta t \cdot |p_l - 1|\, \sigma_h \,, \\ \sigma_p^+ &= \mathcal{O}(1) \cdot \Delta t \cdot |p_l - 1|\, \sigma_h \,. \end{cases}$$

If the initial jump is of the second family, with strength σ_p, we have

$$\sigma_p \qquad \Longrightarrow \qquad \begin{cases} \sigma_h^+ &= \mathcal{O}(1) \cdot \Delta t \cdot h_l\, \sigma_p \,, \\ \sigma_p^+ &= \sigma_p + \mathcal{O}(1) \cdot \Delta t \cdot h_l\, \sigma_p \,. \end{cases}$$

Big picture: Establish following a-priori bounds, (neglecting wave interactions)

(E1). $\|h(t,\cdot)\|_{\mathbf{L}^1}$, $\|p(t,\cdot)-1\|_{\mathbf{L}^1}$: the total amount of mass

(E2). $\inf_x p(t,x)$: a lower bound on the slope

(E3). $\|h(t,\cdot)\|_{\mathbf{L}^\infty}$, $\|p(t,\cdot)\|_{\mathbf{L}^\infty}$: uniform bounds on h and $p-1$;

(E4). TV$\{h(t,\cdot)\}$, TV$\{p(t,\cdot)\}$: bounds on the total variation

Last step: If $\|h\|_\infty$ is small, the additional contribution of interaction to total variation is small.

compactness gives convergence ⇒ completing the proof

Some details on the a priori bounds: define "smart" weighted functionals which are non-increasing in time

For example: bounds on $\|h(t,\cdot)\|_{\mathbf{L}^1}$, $\|p(t,\cdot)-1\|_{\mathbf{L}^1}$:

Write $q = p - 1$, then

$$\begin{aligned} h_t - ((q+1)h)_x &= qh \\ q_t + (hq)_x &= 0 \end{aligned}$$

L^1 bound on q: OK

For h, define the weighted functional:

$$\mathcal{I}^h(t) \doteq \int W(t,x)h(t,x)\,dx\,, \qquad W(t,x) \doteq \exp\left\{\int_{-\infty}^{x} |q(t,y)|\,dy\right\}$$

Transversal wave speeds gives $\frac{d}{dt}\mathcal{I}^h(t) \leq 0$

Initial and boundary value problem:

$$h(0,x) = \bar{h}(x), \quad p(0,x) = \bar{p}(x), \qquad x \leq 0$$

$$p(t,0)h(t,0) = F(t), \qquad t \geq 0$$

Assumptions: same as for (1), in addition, we assume

$$\|F\|_\infty \leq \delta, \quad \mathrm{TV}(F) \leq M$$

Global existence of large BV solutions: (Amadori & Shen, 2008)
The IBVP has a global solution, with uniformly bounded total variation for all $t \geq 0$.

Proof: similar a priori estimates, but needs additional estimates:

– reflecting waves on the boundary $x = 0$
– new waves generated at the boundary due to jumps in F

Both estimates contain either $\|h\|_\infty$ or $\|F\|_\infty$ term.

Limit as $\|h\|_\infty \to 0$ (slow erosion):

study the limiting behavior of mountain shape (slope p).

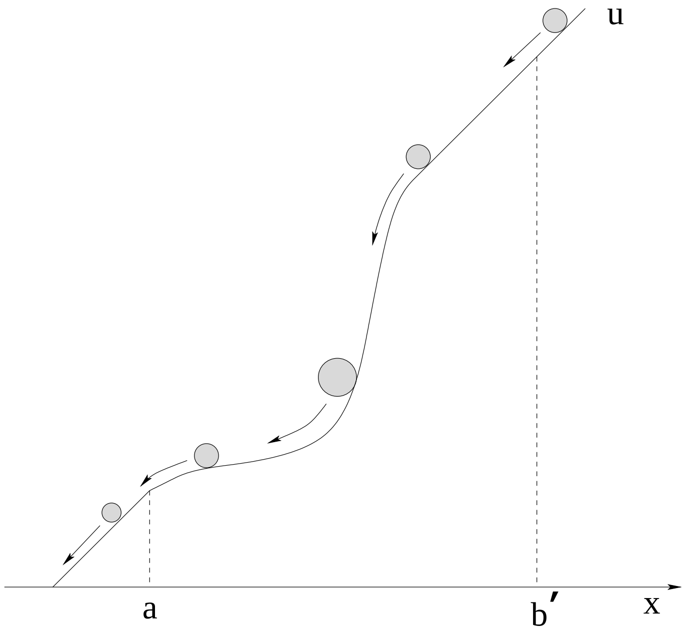
Formal derivation:

Assume the slope p varies very slowly in time.

$\mu(t) = \int_0^t F(t)\,dt$: (amount of mass passing through $x = 0$)

$\Delta\mu$: a small amount of mass pouring down initially from the point $x = 0$.

$\Delta\mu(x(t))$: the size of the avalanche at $x(t) < 0$



$$\frac{\partial}{\partial x}\Delta\mu(x) = \frac{dt}{dx}\cdot\frac{\partial}{\partial t}\Delta\mu(x(t)) = \frac{1}{-p(x)}\cdot(p(x)-1)\Delta\mu(x)$$

(Formal derivation continues:) Hence

$$\Delta\mu(x) = \left(\exp\int_x^0 \frac{p(y)-1}{p(y)}\,dy\right)\Delta\mu(0) \qquad x<0\,.$$

In turn, when this avalanche crosses the point x, it produces a change in the height of the mountain u

$$\Delta u(x) = \frac{1-p}{p}\cdot\Delta\mu(x) = -\left(\frac{p(x)-1}{p(x)}\cdot\exp\int_x^0 \frac{p(y)-1}{p(y)}\,dy\right)\Delta\mu(0)\,.$$

This gives

$$\frac{\partial u}{\partial\mu} = -\left(\frac{p(x)-1}{p(x)}\cdot\exp\int_x^0 \frac{p(y)-1}{p(y)}\,dy\right)\,.$$

Differentiating the above equation w.r.t. x, and recalling $p=u_x$, we obtain a scalar integro-differential conservation law for p:

$$p_\mu + \left(\frac{p-1}{p}\cdot\exp\int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy\right)_x = 0\,. \tag{2}$$

Slow erosion limit: (Amadori & Shen, preprint 2009)

Additional assumptions:

$$F(t) \geq 0, \qquad 0 < M' < \int_0^\infty F(\tau)\,d\tau \leq M$$

Then, as $\|\bar{h}\|_\infty \to 0$ and $\|F\|_\infty \to 0$, the solutions (the p-component) of the avalanche system (1) converge to the unique entropy solution to the scalar integro-differential conservation law (2).

Remark: mountain shape depends only on μ (total mass of moving layer), not on the shape of the avalanche.

Sketch of the proof:

Step 1:
Show the scalar integro-differential equation is well-posed.
This is non-trivial because the flux is a global function.

Need to show: the flow generated by (2) is Lipschitz continuous restricted to the domain of functions satisfying the bounds:

$$\inf_{x<0} p(x,t) \geq p_o > 0, \qquad \text{Tot.Var.} p(\cdot,t) \leq M, \qquad \|p(\cdot,t)-1\|_{\mathbf{L}^1(\mathbf{R}_-)} \leq M$$

$q, \tilde{q}$: two solutions of (2) with initial data $q_0(x), \tilde{q}_0(x)$

One can show that

$$\|q(\tau,\cdot) - \tilde{q}(\tau,\cdot)\|_{\mathbf{L}^1(\mathbf{R}_-)} \leq \|q_0 - \tilde{q}_0\|_{\mathbf{L}^1} + C \cdot \int_0^\tau \|q(s,\cdot) - \tilde{q}(s,\cdot)\|_{\mathbf{L}^1} \, ds$$

Gronwall lemma yields Lipschitz continuous dependence on initial data.

Step 2:
Show the limit of the system, as $\|h\|_\infty$ goes to 0, provides a BV weak solution of the scalar integro-differential equation.
One needs to check the weak (integral) formulation is satisfied in the limit.

Let: q: solution of (1), $\widehat{q}$: limit of q as $\|h\|_\infty \to 0$
must show that: for any fixed $0 \le \mu_1 < \mu_2$ and any test function $\psi \in \mathcal{C}^1_c(\mathbf{R}_-)$,

$$\int_{-R}^{0} \psi(x)\,[q(t(\mu_2),x) - q(t(\mu_1),x)]\,dx = \int_{t(\mu_1)}^{t(\mu_2)} \int_{-R}^{0} \psi_x(x) q(t,x) h\,dx\,dt$$

converges to

$$\int_{-R}^{0} \psi(x)\,[\widehat{q}(\mu_2,x) - \widehat{q}(\mu_1,x)]\,dx = \int_{\mu_1}^{\mu_2} \int_{-R}^{0} \psi_x(x) \left[\frac{\widehat{q}(\mu,x)}{\widehat{q}(\mu,x)+1} \cdot \widehat{k}(\mu,x)\right] dx\,d\mu$$

$$\text{where} \quad \widehat{k}(\mu,x) \doteq \exp \int_{x}^{0} \frac{\widehat{q}(\mu,\zeta)}{\widehat{q}(\mu,\zeta)+1}\,d\zeta.$$

LHS: OK.
RHS: $qh = \dfrac{q}{q+1} \cdot (q+1)h$. We can prove

$$\text{weak convergence of the flux}: \quad (q(\mu,x)+1)\,h(\mu,x) \ \rightharpoonup \ \widehat{k}(\mu,x)$$

$$\text{strong convergence}: \quad \frac{q(\mu,x)}{q(\mu,x)+1} \quad \to \quad \frac{\widehat{q}(\mu,x)}{\widehat{q}(\mu,x)+1}$$

Step 3:
Check entropy admissibility of all shock waves.
One needs to check all shocks jump in the correct direction (jump up).