The Slow Erosion Limit in a Model of Granular Flow

DEBORA AMADORI — University of L'Aquila, Italy
WEN SHEN — Penn State University (USA)

A boundary value problem

We consider the initial-boundary value problem, on the quarter plan $\{(x,t):\ x<0\,, t>0\}$, for the system

$$\begin{cases} h_t - (hp)_x &= (p-1)h\,, \\ p_t + \big((p-1)h\big)_x &= 0\,. \end{cases}$$

with initial data

$$h(0,x) = \bar h(x) \geq 0\,, \qquad p(0,x) = \bar p(x) > 0\,, \qquad x<0$$

and boundary condition

$$h(t,0)p(t,0) \ = \ F(t) \geq 0\,, \quad t>0\,.$$

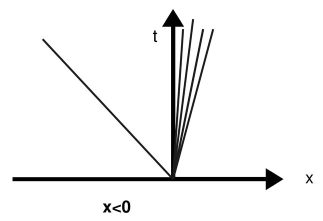
The boundary condition above corresponds to assigning the flux of the h variable: the incoming quantity of h is prescribed by the map $F(t)$.

The system above arises in the study of the flow of granular materials. It is derived from a model introduced by Hadeler & Kuttler. See the References in [1].

Remarks

- The characteristic speeds of this system satisfy: $\lambda_1 < 0 \leq \lambda_2$

  $\Rightarrow$ one single characteristics enters the domain
  $\Rightarrow$ one scalar b.c. is assigned.

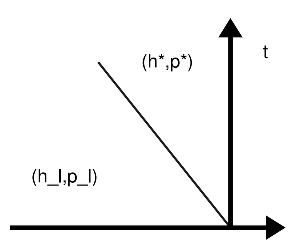


- The quantity assigned at the boundary satisfies

$$\nabla(hp)\cdot r_1(h,p) \neq 0\,,$$

where r_1 is a characteristic eigenvector corresponding to λ_1. This leads to the unique solvability of the boundary Riemann problem and Lipschitz dependence on the boundary data.

at $t=0$: $\quad h_\ell \geq 0,\ p_\ell > 0$
at $x=0$: $\quad F_b \geq 0$
$\Rightarrow$ $\exists!\ (h^*,p^*)$, with $\ h^*p^* = F_b$,
connected to (h_ℓ, p_ℓ) by a 1-wave.



Global existence, half space $x<0$

For some M, $p_0 > 0$ assume that

- $\mathrm{TV}\bar p$, $\mathrm{TV}\bar h \leq M$; $\quad \mathrm{TV}\bar F \leq M$
- $\|\bar h\|_{\mathbf{L}^1}$, $\quad \|\bar p - 1\|_{\mathbf{L}^1} \leq M$; $\quad \|\bar F\|_{\mathbf{L}^1} \leq M$,
- $\bar p(x) \geq p_0 > 0$,

Theorem 1([2]). Given M, $p_0 > 0$ there exists $\delta > 0$ such that if the above assumptions hold and also if

$$\|\bar h\|_{\mathbf{L}^\infty} \leq \delta\,, \qquad \|\bar F\|_{\mathbf{L}^\infty} \leq \delta\,,$$

then the initial-boundary value problem has a global weak solution on $\{x<0\,, t>0\}$, with uniformly bounded total variation for all $t \geq 0$.

Proof: it follows the same lines of the proof for the Cauchy problem (see [1]), taking into account of the boundary effect.

The "slow avalanche" limit

We study the limit of solutions to the initial-boundary value problem as

$$\|\bar h\|_{\mathbf{L}^\infty}\,, \qquad \|\bar F\|_{\mathbf{L}^\infty} \to 0$$

but the total mass $\int_0^\infty \bar F(\tau)\,d\tau$ is uniformly positive.

While h is expected to vanish, the limiting behavior of p occurs to be non-trivial. It can be conveniently analyzed in terms of a new "time" variable. Define the map

$$t \mapsto \mu(t) = \int_0^t F(\tau)\,d\tau \tag{1}$$

that represents the total amount of matter that has flown down between time 0 and t. It is non-decreasing, then admits a (generalized) inverse. One can then re-parametrize the solution in terms of μ as a time variable.

The formal limit

Let us compute the formal limit with $\bar h = 0$ and the following boundary data:

$$(ph)(t,0) \ = \ F^\varepsilon(t) \doteq \varepsilon F^1(\varepsilon t) > 0$$

Then $\|F^\varepsilon\|_{\mathbf{L}^1} = \|F^1\|_{\mathbf{L}^1} > 0$ and $\ \|F^\varepsilon\|_{\mathbf{L}^\infty} \to 0\ $ as $\varepsilon \to 0$. Rescaling the time variable as in (1), one has

$$\frac{\partial}{\partial t} = \varepsilon \tilde F(\mu) \frac{\partial}{\partial \mu}$$

with $\tilde F(\mu) > 0$. The system becomes

$$\begin{cases} \varepsilon\,\tilde F(\mu)\,h^\varepsilon_\mu - (h^\varepsilon p^\varepsilon)_x &= (p^\varepsilon - 1)h^\varepsilon \\ \varepsilon\,\tilde F(\mu)\,p^\varepsilon_\mu + \big((p^\varepsilon - 1)h^\varepsilon\big)_x &= 0\,. \end{cases}$$

As $\varepsilon \to 0$, expect that

$$h^\varepsilon = \mathcal{O}(1)\varepsilon\,, \quad h^\varepsilon_\mu = \mathcal{O}(1)\varepsilon\,, \quad p^\varepsilon = \mathcal{O}(1)\,.$$

It is convenient to set

$$m^\varepsilon \doteq \frac{h^\varepsilon p^\varepsilon}{\varepsilon} \ = \ \mathcal{O}(1)\,.$$

In terms of m^ε, the system rewrites as

$$\begin{cases} \tilde F(\mu)\,h^\varepsilon_\mu - (m^\varepsilon)_x &= \dfrac{p^\varepsilon - 1}{p^\varepsilon} m^\varepsilon\,, \\ \tilde F(\mu)\,p^\varepsilon_\mu + \left(\dfrac{p^\varepsilon - 1}{p^\varepsilon} m^\varepsilon\right)_x &= 0\,. \end{cases} \tag{2}$$

As $\varepsilon \to 0$, one has $\ m^\varepsilon \to m$, $\ p^\varepsilon \to p$ and only the red term in (2) vanishes. Then the first equation reduces to

$$-m_x = \frac{p-1}{p} m$$

Solve this linear equation for m on $x<0$, with boundary condition

$$m(0,\mu) = \tilde F(\mu)\,,$$

in terms of p, and get

$$m(x,\mu) = \tilde F(\mu)\cdot \exp \int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy\,.$$

Substitute in the second eq. of (2) – after having passed to the limit $\varepsilon \to 0$ – and get

$$\tilde F(\mu)\cdot\left[p_\mu + \left(\frac{p-1}{p}\cdot \exp \int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy\right)_x\right] \ = \ 0\,.$$

Up to simplifying $\tilde F$, the result is a closed equation for p!

The limit equation

The formal limit leads to a scalar equation for $p(\mu,x)$:

$$p_\mu + \left(\frac{p-1}{p}\cdot k(\mu,x)\right)_x = 0\,,$$

$$k(\mu,x) = \exp \int_x^0 \frac{p(\mu,y)-1}{p(\mu,y)}\,dy$$

This is a scalar integro-differential conservation law for p on $x<0$. Note that if

- $\mu \mapsto p(\mu,\cdot) - 1 \in \mathbf{L}^1(\mathbb{R}_-) \cap \mathbf{L}^\infty(\mathbb{R}_-)$, $\quad$ with uniform bounds
- $\inf_x p(\mu,x) \geq p_0\quad$ for some $p_0 > 0$, for all μ
- $\displaystyle\int_x^0 |p(t_2,x) - p(t_1,x)|\,dx \leq L|t_2 - t_1| \qquad$ for all x

then the integral term $k(\mu,x)$ is bounded and Lipschitz continuous.
The definition of weak entropic solution for this equation is quite natural.

Passing to the limit

Theorem 2([2]). Let $\bar h_\nu$, $\bar p$, F_ν satisfy the assumptions of Theorem 1, and

$$\|\bar h_\nu\|_{\mathbf{L}^\infty}\,, \qquad \|F_\nu\|_{\mathbf{L}^\infty} \to 0\,, \qquad \nu \to \infty\,,$$
$$M' < \int_0^\infty F_\nu(\tau)\,d\tau \leq M \qquad \forall\, \nu \in \mathbb{N}\,, \tag{3}$$

for some constants $0 < M' < M$. Let $(h_\nu, p_\nu)(t,x)$ be the solution of the initial-boundary value problem as from Theorem 1, and denote by $\tilde p_\nu(\mu,x)$ the p_ν, re-parametrized in terms of μ. Then, as $\nu \to \infty$,

$$\tilde p_\nu(\mu,x) \to P(\mu,x) \text{ in } \mathbf{L}^\infty\Big([0,M']\,;\ \mathbf{L}^1(\mathbb{R}_-)\Big)$$

where P is an entropy solution to

$$\begin{cases} P_\mu + \big(f(P)\cdot k(\mu,x)\big)_x = 0\,, \\ P(0,x) = \bar p(x) \end{cases}$$

for $x<0$, $\mu \in [0,M']$, and

$$f(p) = \frac{p-1}{p}\,, \qquad k(\mu,x) = \exp \int_x^0 f(P(\mu,y))\,dy\,.$$

Remarks

- The initial data for p does not depend on ν.
- No boundary condition at $x=0$ is specified for

$$P_\mu + \big(f(P)\cdot k(\mu,x)\big)_x = 0\,.$$

Indeed, the characteristic speed $f'(P)$ is non-negative (for $P>0$) and $k(\mu,x) > 0$.

- The limiting profile $P(\mu,\cdot)$ is defined on the interval $[0,M']$, see (3) above. It does not depend on the pointwise values of the fluxes F_ν.
  In essence, the limiting profile $P(\mu,\cdot)$ depends only on the total mass of h that has flown down, but not on its specific rate.

References

[1] Amadori, D. and Shen, W.; Global Existence of Large BV Solutions in a Model of Granular Flow. `http://www.math.ntnu.no/conservation/2008/023.html` accepted for publication on *Comm. PDE*

[2] Amadori, D. and Shen, W.; The Slow Erosion Limit in a Model of Granular Flow `http://www.math.ntnu.no/conservation/2009/003.html`